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AN ELEMENTARY TREATISE

ON

DYNAMICS.

FOURTH EDITION.

AN ELEMENTARY TREATISE

ON

THE INTEGRAL CALCULUS,

CONTAINING

APPLICATIONS TO PLANE CURVES AND SURFACES.

BY

BENJAMIN WILLIAMSON, F.R.S.

FIFTH EDITION.

AN ELEMENTARY TREATISE

ON

THE DIFFERENTIAL CALCULUS,

CONTAINING

THE THEORY OF PLANE CURVES.

BY

BENJAMIN WILLIAMSON, F.R.S.

AN ELEMENTARY TREATISE

ON

DYNAMICS,

CONTAINING

APPLICATIONS TO THERMODYNAMICS,

WITH

NUMEROUS EXAMPLES.

BY

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PREFACE.

ALTHOUGH in recent years several most important works on Dynamics have been published in England, yet none have been issued which seem to fill the *rôle* contemplated in this book. In its composition we have started from the most elementary conceptions, so that any Student who is acquainted with the notation of the Calculus can commence the treatise without requiring the previous study of any other work on the subject. The first half contains a tolerably full treatment of what is usually styled the Dynamics of a Particle. The latter half treats of the Kinematics and Kinetics of Rigid Bodies; and throughout we have kept the practical nature of the subject in view, and have, in general, avoided purely fancy problems.

In an early chapter we have introduced and elucidated the general principle of Work or Energy, and have given subsequently a more complete treatment of this great principle, illustrating it by a brief application to the theory of Thermodynamics. In the latter part of the book we have borrowed largely from Thomson and Tait's *Natural Philosophy;* Routh's *Rigid Dynamics;* Schell's *Theorie der Bewegung und der Kräfte;* and Clausius' *Mechanical Theory of Heat;* our aim having been simply to enable the Student to acquire as easily as possible a knowledge of the subject of which we treat.

To those who desire to pursue the study of Dynamics to its highest development, the perusal of the great treatise of Thomson and Tait, as also that of Routh, will, we hope, be facilitated by using the present work as an introduction.

We cannot conclude without expressing our deep obligations to Mr. CATHCART, for most valuable assistance during the entire progress of the book.

TRINITY COLLEGE,
January, 1885.

CORRIGENDA.

Page 63, last line, *for* $\frac{R+Q-R}{P+Q+R} \cdot \frac{P+Q}{P-Q} s$, *read* $\frac{P+Q-R}{P+Q+R} \cdot \frac{P+Q}{P-Q} s$.

,, 69, second line from bottom, *for* place, *read* plane.

,, 84, eighth line from bottom, *for* $\cos \frac{1}{2} \epsilon - \epsilon'$, *read* $\cos \frac{1}{2} (\epsilon - \epsilon')$.

,, 244, fifth and seventh lines from the bottom, *for* the areal, *read* twice the areal.

J. A. Workman,
Q. C. C.
January 1888.

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DYNAMICS.

CHAPTER I.

VELOCITY.

1. **Matter.**—We give the name of matter to that which we regard as the permanent cause of any of our sensations. Portions of matter which are bounded in every direction are called bodies. Every body has necessarily a determinate volume, and an external form or surface; and exists, or is conceived to exist, in space.

A portion of matter indefinitely small in all its dimensions is called a material *particle.* Every body of finite dimensions may be regarded as consisting of an indefinitely great number of particles. The name of *force* is given to any cause which produces, or tends to produce, motion in matter. The branch of Mechanics which treats of motion produced in a body by the action of force is commonly called Dynamics. We commence with the consideration of motion in itself, without any regard to its cause.

2. **Motion, Velocity.**—When a body continually changes its position in space, it is said to be in motion; and the *rate* at which any of its points is moving at any instant is called the *velocity* of the point at that instant.

The motion of a point is said to be *rectilinear* or *curvilinear*, according as its path is a right line or curved.

In the case of curvilinear motion, the *direction* of motion of a particle at any instant is that of the tangent to its path, drawn at the point occupied by the moving particle at the instant.

3. **Motion of Translation.**—If all the points of a rigid body move, at each instant, in parallel directions, the body is said to have a motion of translation only; and the motion of the body is completely determined when that of any one of its points is known. It is usual, in this case, to take its centre of mass as the point whose motion determines that of the body.

In our earlier chapters whenever we speak of a rigid body moving, we shall suppose it to have a motion of translation solely, and we consider its path as that of its centre of mass.

4. **Uniform Motion, Velocity.**—If a point move over equal lengths or spaces,* in equal intervals of time, however short the intervals be taken, its motion is said to be uniform; and its velocity is measured by the *space described in the unit of time:* this is the same at every instant so long as the motion continues uniform.

A *second* is usually adopted as the unit of time; and, in this country, a foot as the unit of length. Thus, the velocity of a point which moves over five feet in each second is said to be a velocity of 5 feet per second, and is numerically denoted by 5; and similarly in other cases. If any other units of time and space be adopted, the number which represents the velocity of the moving point will have to be altered proportionally. Thus, we speak of a velocity of 10 miles an hour, or 100 yards a minute, &c.: each of these can be readily expressed in feet per second, when necessary.

The space, or length of the path described during any time, is usually denoted by the letter s, the velocity by v, and the time estimated in seconds by t.† In the case of *uniform motion*, the relation connecting these quantities can be immediately obtained. For, if the space described in one second be represented by v, that described in two seconds is represented by $2v$, that in three seconds by $3v$, and that in any number (t) of seconds by vt.

---

* The word *space* is employed in abbreviation for *length of path* described.

† Unless the contrary be stated, we shall in all cases assume a foot and a second as our units of space and time, *i.e.* we shall regard t as representing a *number* of seconds or parts of a second, and s as a number of feet.

Accordingly we have in the case of uniform motion the relation

$$s = vt. \tag{1}$$

This formula evidently holds whatever be the *units* of space and time, and introduces the unit of velocity as that of a unit of space described in a unit of time. It is true for uniform curvilinear, as well as rectilinear motion; and also whether t represents a number of seconds, or any part of a second, however small.

Again, if s' denote the space described in the time t', we have $s' = vt'$, and hence

$$v = \frac{s' - s}{t' - t};$$

or the velocity, when uniform, is measured by the *space described during any interval of time divided by the number by which that time is represented.*

This result equally holds if we suppose the interval of time, denoted by $t' - t$, to become *indefinitely* small; in which case the *limiting value* of $\dfrac{s' - s}{t' - t}$ or $\dfrac{ds}{dt}$ will still represent the velocity v.

EXAMPLES.

1. If a body, moving uniformly, pass over 10 miles in an hour, find its velocity in feet per second. *Ans.* $14\frac{2}{3}$.

2. If a body, moving uniformly with a velocity of 16 feet, pass over 100 miles, find the time of its motion. *Ans.* 9 hrs. 10 min.

3. Assuming that light travels from the sun to the earth in $8^{\text{m}}\ 30^{\text{s}}$, and that its velocity is 180,000 miles, calculate the distance of the sun. *Ans.* 91,800,000 miles.

4. If a velocity of 20 miles an hour be the unit of velocity, and a mile the unit of space, find the number which represents a velocity of 32 feet per second. *Ans.* $1\frac{1}{11}$.

5. Find in metres the velocity of a point on the earth's equator arising from the rotation of the earth on its axis. *Ans.* 463.

5. **Variable Motion.**—If the spaces described in equal intervals of time be not equal, the motion is said to be *variable*, and the velocity can no longer be measured by the space actually described in one second. The moveable has, however, at each instant a certain definite velocity which is

measured by the space *which it would describe during a second, if it were conceived to move uniformly during that time with the velocity which it has at the instant under consideration.*

For example, when we say that a railway train is moving at the rate of 40 miles an hour, we mean that it would pass over 40 miles in the hour if it continued to move during that time with the speed which it has at the instant referred to.

Again, if we suppose that there are no *sudden* changes of velocity, the change in the velocity of a moveable in any indefinitely small portion of time must be itself indefinitely small; as otherwise the velocity would not vary *continuously*. Accordingly, in such cases, we may suppose the motion as *remaining uniform* during the indefinitely small time dt; and we shall have (as in the last Article) for the velocity v at any instant the equation

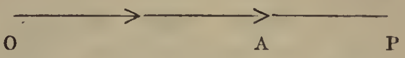
$$v = \frac{s' - s}{t' - t} = \frac{ds}{dt}. \tag{2}$$

That is, in all cases the velocity of a point at any instant is measured by the limiting value of the space described in a small interval of time, divided by the number which represents that interval of time. This method of expressing velocity is sometimes concisely represented in the notation of Newton by the symbol $\dot{s}$.

6. **Mean Velocity.**—If a body describe the space s in the time t, then its *mean* or *average* velocity during that time is represented by $\frac{s}{t}$, being the velocity with which a body, moving uniformly, would describe the same space in the time t. The formula (2) can be immediately deduced from the consideration of mean or average velocity—for we may consider the velocity of a point at any instant as being its mean velocity during an infinitely small interval of time; whence we get, as before, the relation $v = \frac{ds}{dt}$.

7. **Geometrical Representation of a Velocity.**—Uniform rectilineal motion is completely determined when the direction and the velocity, or rate of motion, are known. Hence the velocity of a point can be represented both in magnitude and direction by a right line.

Thus, if a point move uniformly in the line OP, so as to describe the space OA in the unit of time (one second suppose), the line OA may be taken to represent the velocity of the point both in magnitude and direction. The *arrow head* denotes the direction in which the motion takes place namely, from O to A.



This method of representation equally holds in the case of variable velocity, provided OA be the space which the body *would* describe in one second, if its velocity remained unaltered in magnitude and direction (Art. 5).

In accordance with the principles established in Geometry, if the velocity of a particle moving from O to P be regarded as *positive*, velocity in the opposite direction, *i. e.* from P to O, must be regarded as *negative*.

8. **Kinematics.**—As our ideas of motion and velocity depend solely on our elementary conceptions of space and time, the whole subject of motion admits of being treated as a branch of pure Mathematics; and, as such, has been discussed in many important treatises during recent years.

This branch of Mathematics is called Kinematics* (from *κίνημα*, motion), and in it the motion of a body is discussed without any reference to the force or forces by which the motion is produced. Questions of the latter class, *i. e.* of motion with reference to force, belong to the science of Dynamics, or what is now frequently styled Kinetics.

The foregoing distinction should be observed by the student, as much indistinctness of conception arises from its not being carefully kept in mind in the study of Dynamics.

In the present treatise it is not however proposed to divide the treatment of the subject in the manner indicated, as to do so would require a complete discussion of motion (including rotation and kindred subjects) before entering on the most elementary problems in Dynamics. At the same time it will aid the student towards obtaining clear mechanical conceptions if he will consider what part of each problem discussed belongs properly to the science of Kinematics, and what to that of Dynamics or Kinetics.

---

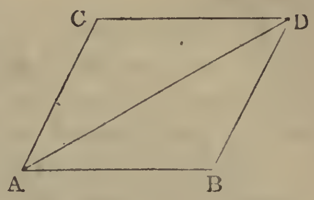
* The name "Cinématique" was first given to this branch of Mathematics by Ampère, in his "Essai sur la philosophie des Sciences," 1834.

9. **Rest and Motion, Relative.**—We have defined rest and motion with reference to space. Now of space in itself or absolute space our senses take no cognizance, all that we perceive being matter or body as occupying or existing in space; but our senses give us no information as to whether any body occupies the same absolute position in space during successive intervals of time or not. Hence, of absolute rest we can have no perception or knowledge; and when we say that a body is at rest we mean that it does not alter its position with relation to other bodies which are considered fixed. For instance, bodies on the earth's surface are said to be at rest when they do not alter their position relatively to the earth's surface, supposed fixed; we know however that the earth has at least two distinct motions, one of rotation relative to its axis; the other around the sun, regarded as fixed. As our idea of rest is only relative, so also must be our idea of motion: thus, a body is said to be in motion when it alters its position with respect to other bodies regarded as being at rest.

Hence all motions must be considered as relative: for instance, when we say that a body is moving at the rate of thirty miles an hour, we mean that such is its velocity relative to a place on the earth: its absolute velocity is immensely greater, and is obtained by combining this velocity with the absolute velocity of the earth itself.

Again, we speak of the same body as being at *rest*, or as being in *motion*, according as we compare its position with that of one object or of another. For example, a person seated in a railway carriage is said to be at rest relatively to the carriage, and to be in motion relatively to the earth, &c.

That a body may be regarded as having at the same instant two or more velocities is a matter of common experience: for instance, if a ball roll along the deck of a vessel, which is descending a river, we conceive the ball as having simultaneously one velocity along the deck; another, that of the vessel in the stream; a third, that of the river relatively to its banks, &c. The velocity of the ball, relatively to the earth, is got by compounding these separate velocities. We proceed to show in what manner this can be done.



10. **Composition of Velocities.**—Suppose a point to move uniformly, with a velocity v, along the line AB, while the line moves uniformly parallel to itself; then the point may be regarded as animated by the two velocities simultaneously. In order to find its position at the end of any time t, we take AB to represent the space which it would describe in the time t, along AB considered as fixed; and let CD be the position of the moving line at the end of the same time; complete the parallelogram $ABDC$; then D will plainly represent the position of the moving point at the end of the time t. Also, if v' be the uniform velocity of the point along the line AC, we shall have $AC = v't$, and $CD = vt$. Hence

$$\frac{AC}{CD} = \frac{v'}{v}.$$

Again, as this is independent of t, the ratio of AC to CD will be constant during the entire motion; and consequently the point will move from A to D along the diagonal AD.

To find the velocity of the moving point, we make $t = 1$ (or the unit of time) in the last; then AB and AC represent in magnitude and direction the component velocities of the moving point, and AD represents the resultant velocity: in other words, if a body be animated by two velocities represented in magnitude and direction by the sides of a parallelogram, the resultant velocity is represented in magnitude and direction by the diagonal of the parallelogram.

Conversely, any velocity may be regarded as equivalent to two velocities in any two directions, and the magnitudes of the component velocities can be determined by the preceding.

In like manner, if a body be animated simultaneously with three velocities, its resultant velocity is represented in magnitude and direction by the diagonal of the parallelepiped whose edges represent the component velocities. For we can compound two of these velocities by the method given above, and then compound their resultant with the third velocity. This principle can, plainly, be extended to the case of a point supposed to be animated by any number of velocities simultaneously.

11. **Polygon of Velocities.**—It immediately follows that if a point be subjected to any number of simultaneous velocities its resultant velocity can be obtained by the following geometrical construction:—

From O, the original position of the point, draw OA, representing one of the given velocities in magnitude and direction; from A draw AB, parallel and equal to the line which represents a second velocity; and so on for the remaining velocities; then the line which connects O with the extremity of the line drawn parallel and equal to the line representing the last velocity will represent the resultant velocity, both in magnitude and direction.

This construction is called the polygon of velocity, and is in general a *gauche* polygon.

The preceding result admits of being stated otherwise, thus: If a body be subjected to two or more uniform velocities it will arrive at the same position at the end of any time as it would have arrived at if the several motions had taken place *successively* instead of simultaneously. This is adopted as an axiom by some writers on Mechanics, for it appears to be an immediate consequence of our ideas of motion. The student can easily see that the whole theory of the composition of velocities can be deduced from this principle.

12. **Component and Resultant Velocities.**—The velocities represented by AB and AC, in Art. 10, are called the *components* of the velocity represented by AD.

If a point describe any plane curve, the usual method of representing its position is with reference to two fixed rectangular axes lying in the plane.

Then, if x, y be the coordinates of the moving point at any instant, its component velocities, respectively, parallel to the coordinate axes, are evidently, by Art. 5, represented by $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Also, if α be the angle which the direction of motion at the instant makes with the axis of x, the component velocities are represented by $v \cos \alpha$ and $v \sin \alpha$, respectively; *i.e.* the velocity with which a point is moving in any fixed direction is equal to the component of its velocity in that direction.

Hence we get $v\cos a = \frac{dx}{dt}, \quad v\sin a = \frac{dy}{dt}.$ (3)

If we square and add, we get

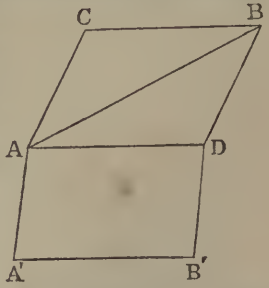
$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2; \therefore v = \frac{ds}{dt};$$

i.e. the velocity in a curvilinear path is represented in the same matter as in a rectilinear; this result might also have been directly established from other considerations. More generally, if x, y, z be the coordinates of a moving point at any instant, with reference to any system of coordinate axes, its component velocities parallel to the coordinate axes are plainly represented by $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$, respectively. If the axes be rectangular, and if a, β, γ be the direction angles of the line of motion of the point, and v its velocity, then the component velocities parallel to the coordinate axes are represented by $v\cos a$, $v\cos\beta$, $v\cos\gamma$, respectively. Hence, in this case, we have

$$v\cos a = \frac{dx}{dt}, \quad v\cos\beta = \frac{dy}{dt}, \quad v\cos\gamma = \frac{dz}{dt}. \tag{4}$$

In Newton's notation, as in Art. 5, these component velocities are represented by the symbols, $\dot{x}$, y, $\dot{z}$.

13. **Relative Velocity.**—If the point A be in motion along AB with a velocity represented by AB, and, at the same time, A' be in motion along $A'B'$ with a velocity represented by $A'B'$, to find their *relative* velocity.



Draw AD parallel and equal to $A'B'$, and construct the parallelogram $ACBD$; then the velocity AB may be regarded as equivalent to the velocities AD and AC; now the former velocity, being *equal and in the same direction as that of the other point* A', *will not alter the relative position of the points* (Art. 10); consequently the latter component AC represents the relative velocity of the moving points, *i.e.* the velocity with which A is moving relatively to A', regarded as at rest.

Hence, to get the velocity of one moving point relatively to another, which is also in motion, we suppose *equal and parallel motions given to both, each equal and opposite to the motion of the second point:* by this means that point is brought to rest, and the velocity of the other, relative to it, is had by compounding the new velocity with its original velocity.

14. **Components of Relative Velocity.**—Suppose (x, y, z), (x', y', z') to be the coordinates of the two moving points (M, M'), respectively, with reference to any coordinate system of fixed axes. Then, to get the motion of M', relatively to M, we suppose three axes drawn through M parallel, respectively, to the coordinate axes; and let ξ, η, ζ denote the coordinates of M, relative to these axes, and we have

$$\xi = x' - x, \quad \eta = y' - y, \quad \zeta = z' - z;$$

and hence

$$\frac{d\xi}{dt} = \frac{dx'}{dt} - \frac{dx}{dt}, \quad \frac{d\eta}{dt} = \frac{dy'}{dt} - \frac{dy}{dt}, \quad \frac{d\zeta}{dt} = \frac{dz'}{dt} - \frac{dz}{dt}; \qquad (5)$$

or

$$\frac{dx'}{dt} - \frac{dx}{dt}, \quad \frac{dy'}{dt} - \frac{dy}{dt}, \quad \frac{dz'}{dt} - \frac{dz}{dt}$$

represent the components of the relative velocity of the two moving particles.

EXAMPLES.

1. Two points are moving in rectangular directions, with velocities of 300 and 400 yards per minute; find their relative velocity in feet per second.
Ans. 25.

2. Two particles start simultaneously from different points, in given directions, with uniform velocities. Show how, by a geometrical construction, to determine the relative distance at the end of any time; and find when this distance is a minimum.

3. The tide is running out of the mouth of a harbour at the rate of $2\frac{1}{2}$ miles per hour; in what direction must a man, who can row in still water at the rate of 5 miles per hour, point the head of the boat in order to make for a point directly across the harbour?

4. A boat starts with a given velocity across a river; find the direction in which she should steer, in order, without altering her course, to land at a given station at the opposite side of the river—the velocity of the stream, and also of the boat, being supposed known.

5. Two trains are moving, one due south, the other north-east. If their velocities be 25 and 30 miles an hour, respectively, calculate their relative velocity.

6. A railway train is moving at the rate of 30 miles an hour, when it is struck by a stone moving horizontally and at right angles to the train with the velocity of 33 feet per second. Find the magnitude and direction of the velocity with which the stone appears to meet the train.
Ans. Resultant velocity is 55 feet.
Indian Civil Service Exam., 1876.

7. Two particles start simultaneously from A, B, two of the angular points of a square $ABCD$, in the directions AB, BC, and describe the periphery with constant velocities V, v, respectively, where V is greater than v, until one particle overtakes the other. Prove that the minimum distances between the particles occur at equal intervals of time, and that if $V : v :: m + 1 : m$, where m is an integer, the sum of all these minimum distances is

$$\frac{m(m+1)}{2\sqrt{m^2 + (m+1)^2}} \times \text{a side of the square.}$$

Camb. Math. Trip., 1871.

CHAPTER II.

ACCELERATION.

15. **Acceleration and Retardation of Motion.**—The velocity of a moving body is said to be accelerated or retarded according as it increases or diminishes with the time. This acceleration, or rate of change of velocity, may be either uniform or variable. Retardation of motion is to be regarded as a *negative acceleration*, i.e. as an acceleration in the opposite direction to that of the motion.

16. **Uniform Acceleration.**—The motion of a body is said to be uniformly accelerated when it receives *equal increments of velocity in equal times.* In this case the acceleration is measured by the additional velocity received in each unit of time. As a second is usually taken as the unit of time, we may define the acceleration of velocity in this case to be measured by the *additional velocity received by the moveable in each second;* this acceleration is usually denoted by the letter f.

In the case of uniform acceleration in a right line we proceed to find expressions for the velocity at the end of any given time, and also for the space described.

17. **Velocity at any Instant.**—Let v_0 denote the velocity at the instant from which the time is reckoned; then, since the body receives in each second an additional velocity f, its velocity at the end of the first second is $v_0 + f$; at the end of the next second, $v_0 + 2f$; at the end of the third, $v_0 + 3f$; and at the end of n seconds, $v_0 + nf$. Or, if t denote the number of seconds in question, and v the velocity at the end of that time, we have

$$v = v_0 + ft. \qquad (1)$$

If the body be supposed to start from rest, we have

$$v = ft;$$

that is, *the velocity acquired at the end of t seconds is t times that acquired at the end of one second.*

In the case of a uniformly retarded motion, f denotes the velocity *lost* in each second; and, if v_0 be the initial velocity, we shall have, as before, for the velocity at the end of t seconds,

$$v = v_0 - ft. \tag{2}$$

In this case the velocity becomes zero at the instant when $v_0 = ft$, or at the end of the time $\frac{v_0}{f}$. If the retardation continued afterwards, the velocity would become negative; that is, the body should proceed to move back in a direction opposite to that of its former motion.

It will be observed that the formulæ (1) and (2) differ only in the sign of f; they may accordingly be regarded as comprised in the same general formula, in which a retardation, as stated before, is regarded as a negative acceleration.

Examples.

1. If a body start from rest with a uniform acceleration of 7 feet; find its velocity at the end of three minutes.

Ans. 1260 feet.

2. In what time would a body acquire a velocity of 100 feet if it start from rest with a uniform acceleration of 32 feet?

Ans. $3\frac{1}{8}$ seconds.

3. A body starts from rest with a velocity of 1000 feet, and its motion is uniformly retarded by a velocity of 16 feet each second; find when it would be brought to rest.

Ans. 1 m. $2\frac{1}{2}$ sec.

4. A velocity of one foot per second is changed uniformly in one minute to a velocity of one mile per hour. Express numerically the rate of change when a yard and a minute are taken as the units of space and time.

Ans. $\frac{28}{3}$.

18. **Space described in any Time.**—To find the space described in any time in the case of uniform acceleration in a straight line.

From equation (2) we get

$$\frac{ds}{dt} = v = v_0 + ft;$$

hence, by integration,

$$s = v_0 t + \tfrac{1}{2} ft^2;$$

no constant being added, since the space is measured from the position of the body when $t = 0$.

If the body start from rest, we have

$$s = \tfrac{1}{2}ft^2.$$

In the case of uniformly retarded motion we have

$$s = v_0 t - \tfrac{1}{2}ft^2.$$

This and the preceding formula are represented by the single expression

$$s = v_0 t \pm \tfrac{1}{2}ft^2, \qquad (3)$$

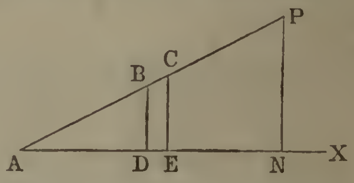
in which the upper or lower sign is given to f, according as the acceleration has place in the positive or negative direction.

Similarly, equations (1) and (2) are combined in the statement

$$v = v_0 \pm ft. \qquad (4)$$

The preceding result admits also of being established geometrically, in the following manner, as given by Newton:—

Suppose the point to start from rest, and on any right line AX take portions AD, AE, &c., proportional to the times from the commencement of the motion, and erect perpendiculars DB, EC, &c., representing the corresponding velocities; then since the velocity at the end of any time (Art. 18) is proportional to that time, the ordinates BD, CE, &c., will be to one another in the same ratio as the times, *i.e.* as AD, AE, &c.; and consequently the points A, B, C, &c., all lie on a right line.



Again, let $AD = t$, $DE = \Delta t$, $BD = v$; then the space described in the infinitely small time Δt will be represented by $v\Delta t$, i. e. by the area $BDEC$; and accordingly the whole space described in the time represented by AN will be represented by the sum of the elementary areas, $BDEC$, &c., or by the whole area, APN, i. e. by $\tfrac{1}{2}AN \times PN$, or by $\tfrac{1}{2}vt$; therefore $s = \tfrac{1}{2}ft^2$, as before.

If the body be supposed to start with an initial velocity v_0, the student will find no difficulty in supplying the corresponding construction.

This result can also be deduced immediately from the consideration of *mean* or *average velocity*. For, since the velocity increases uniformly with the time, the velocity at any time *before* the middle of the interval t must be as much *less* than the velocity at the middle of the interval, as the velocity at the same time *after* the middle of the interval is *greater* than the velocity at the middle of the interval. Hence the velocity at the middle of the interval is the average velocity during that interval, and is therefore equal to the arithmetic mean of the initial and final velocities. Thus

$$\text{average velocity} = \tfrac{1}{2}(v_0 + v_0 + ft) = v_0 + \tfrac{1}{2}ft;$$

therefore
$$\frac{s}{t} = v_0 + \tfrac{1}{2}ft,$$

or
$$s = v_0 t + \tfrac{1}{2}ft^2.$$

19.—**Relation between Velocity and Space.**—If we eliminate t between equations (3) and (4), we get

$$v^2 = v_0^2 \pm 2fs, \qquad (5)$$

in which the upper or lower sign is taken according as the acceleration is in the direction of the motion or in the opposite direction.

We shall resume the consideration of these equations when we come to the investigation of the motion of a body under the action of a constant force.

20. **Algebraic Expression for an Acceleration.**—In the case of a point moving with a uniform acceleration, let v represent the velocity at the end of the time t, and v' that at the time t'; then by (1) we have

$$v = v_0 + ft, \quad v' = v_0 + ft',$$

and hence
$$f = \frac{v' - v}{t' - t}.$$

Moreover, since this result holds, however small the interval of time represented by $t' - t$ may be, we have, as in Art. 4,

$$f = \frac{dv}{dt}.$$

21. **Variable Acceleration.**—In the case of motion of a point in a right line, if the acceleration is not uniform, but varies continuously according to any law, we plainly (as in Art. 5) may suppose that the motion is uniformly accelerated during an infinitely small time dt; or (which is the same thing) that the acceleration at any instant is measured by what the *increase of velocity in a unit of time would have been if its rate of increase had been uniform during that time, and the same as that at the instant in question.* Hence the acceleration at any instant is defined as *the rate of change of the velocity at that instant*, and is measured in all cases by the ratio of the increment of the velocity at the instant to the increment of the time.

Accordingly we have, whether the acceleration be uniform or variable, the relations

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2}, \tag{6}$$

which are expressed in Newton's notation in the form

$$f = \dot{v} = \ddot{s}.$$

All these results apply equally to the case of retardation of motion, which is always to be regarded as a negative acceleration.

22.—**Geometrical Representation of an Acceleration.**—From the preceding it appears that the acceleration of the motion of a point, whether it be uniform or variable, is in all cases measured by a velocity. Hence it can be *represented, both in magnitude and direction, by a right line*, in the same manner as any other velocity (Art. 7).

Hence, also, we may regard a body as receiving two or more simultaneous accelerations of motion, and can determine the resultant acceleration by a geometrical construction, as in Arts. 10 and 11.

Consequently, accelerations are compounded and resolved according to the same laws as velocities. This will also appear as a consequence of the Second Law of Motion, which shall be stated and explained in the next Chapter.

23. **Component Accelerations Parallel to Fixed Axes.**—If x, y, z denote the coordinates relative to a fixed

rectangular system of axes, of the position of a moving point at the end of the time t; then, as in Art. 12, its component velocities parallel to the axes of coordinates are represented by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, respectively.

Hence, since the acceleration of motion in any direction is measured by the rate of change of the velocity in that direction, we have for the accelerations parallel to the axes of coordinates the expressions

$$\frac{d\left(\frac{dx}{dt}\right)}{dt}, \quad \frac{d\left(\frac{dy}{dt}\right)}{dt}, \quad \frac{d\left(\frac{dz}{dt}\right)}{dt},$$

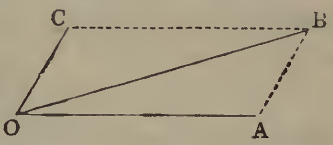
or

$$\ddot{x} = \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}, \quad \ddot{z} = \frac{d^2z}{dt^2}, \tag{7}$$

where, in accordance with Newton's notation, $\ddot{x}, \ddot{y}, \ddot{z}$ denote the accelerations parallel to the axes of x, y, z, respectively. The total acceleration of the motion of the particle is the resultant of these accelerations.

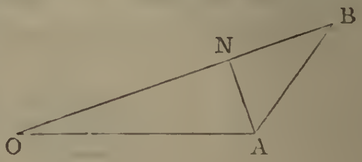
The question of acceleration in curvilinear motion can also be treated in another manner, as follows:—

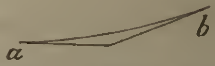
24. **Curvilinear Motion, Change of Velocity, Total Acceleration.**—Suppose a point to move in a curvilinear path, and from any point O, let the line OA be drawn, representing in magnitude and direction the velocity of the moving point at any instant. Let OB, in like manner, represent its velocity at the end of the interval of time Δt. Join AB, and complete the parallelogram $OABC$. Then the velocity represented by OB is equivalent to the component velocities represented by OA and OC; but if the velocity of the point had not changed during the interval Δt, it would have been represented by OA; hence OC, or AB, *represents in magnitude and direction the change of velocity in the time* Δt.



Again, since the acceleration of the velocity of a moveable, at any instant is, in all cases, measured by the *rate of* *change of the velocity* for that instant, it follows, as in (5), that if we regard the interval of time Δt as becoming infinitely small, the acceleration of the motion is represented by the limiting value of $\frac{AB}{\Delta t}$. This limiting value is called the *total acceleration* of the motion of the particle at the instant.

25. **Tangential and Normal Accelerations.** — Again, suppose a to denote the position of the moving particle at the end of the time t, and b its position after a small interval of time, Δt, and draw tangents to the path at the points a and b. Also, as before, from any point O draw OA, OB, parallel to these tangents, and representing the velocities at a and b, respectively. Then, by the preceding Article, AB represents the total change in the velocity in the interval Δt.





Draw AN perpendicular to OB, and suppose the velocity AB resolved into the two, AN and BN; then, the former represents the change of velocity in the normal direction, and the latter in the tangential.

The corresponding accelerations are represented by the limiting values of $\frac{AN}{\Delta t}$ and $\frac{BN}{\Delta t}$, respectively.

Again, let the angle BOA, or the angle between the tangents at a and b, when indefinitely small, be denoted by $d\phi$, and we have

$$AN = OA\,d\phi = v d\phi.$$

The normal acceleration is therefore

$$v\frac{d\phi}{dt} = v\frac{d\phi}{ds}\frac{ds}{dt} = v^2\frac{d\phi}{ds} = \frac{v^2}{\rho} \text{ (\textit{Diff. Calc.}, Art. 226)}, \quad (8)$$

where ρ represents the radius of curvature of the path at the point a.

Also in the limit we have $\frac{BN}{\Delta t} = \frac{dv}{dt}$. Hence the tangential acceleration is represented by $\frac{dv}{dt}$; as is also easily seen from equation (6).

In the case of uniform motion in a circle, since the velocity v is constant, the tangential acceleration vanishes, and the normal acceleration (which then becomes the total acceleration) is $\frac{v^2}{r}$, or $\frac{4\pi^2 r}{T^2}$, where r denotes the radius of the circle, and T the time in which the circle is described.

The normal acceleration in this case is called the *centripetal acceleration*, as it is constantly directed towards the centre of the circle.

26. **Hodograph.***—In accordance with the method of the preceding Articles, if from any point O lines OA, OB, OC, &c., be drawn representing, in magnitude and direction, the velocities at the points, a, b, c, &c., taken consecutively in the path of a particle, then the system of points A, B, C, &c., will lie on a new curve called the hodograph of the original curve, which is considered to be described by A as the point a moves along the given curve.

Since the lines AB, BC, &c., become ultimately tangents to the hodograph, it follows that the direction of the total acceleration at any point a is parallel to the tangent to the hodograph at the corresponding point A.

Also, since the amount of the total acceleration is measured by the limiting value of $\frac{AB}{\Delta t}$, it follows that the total *acceleration, at any point a, is represented by the velocity at the point A in the hodograph.*

We shall give some applications of this method subsequently, more especially in connexion with the treatment of Central Forces.

27. **Angular Velocity, Angular Acceleration.**—If the position of a point P moving in a plane be taken in polar coordinates, r and θ, with reference to a fixed origin O, then the rate of increase of the angle θ is called the angular velo-

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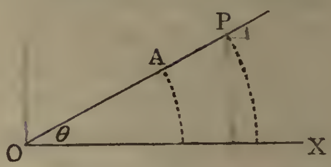
* Sir W. R. Hamilton employed this name (ὁδὸν γράφειν) (*Proceedings*, R. I. A., 1846, p. 344) in his method of discussing the connexion between acceleration and motion. The hodograph is called the *curve of accelerations* by French writers on Mechanics.

city of P relative to the fixed point O. Hence, if ω denote the angular velocity at any instant, we have $\omega = \frac{d\theta}{dt}$.



If OP revolve uniformly, completing its revolution in T seconds, then its angular velocity, in circular measure, is obviously given by the equation

$$\omega = \frac{2\pi}{T}.$$

Suppose OA taken equal to the unit of length, then the velocity of the point A, in its circular path, represents the angular velocity of the line OP.

Again, if the angular velocity of P be variable, its rate of increase is called its angular acceleration; hence the angular acceleration of P with regard to O is represented by $\frac{d\omega}{dt}$ or $\frac{d^2\theta}{dt^2}$.

28. **Accelerations along and perpendicular to the Radius Vector.**—Let x, y be the rectangular coordinates of the moving point P, and r, θ the corresponding polar coordinates, at the end of the time t; then $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ (Art. 23) represent the accelerations parallel to the axes; hence, by Art. 22, the acceleration P along the radius vector is

$$\cos\theta\,\frac{d^2x}{dt^2} + \sin\theta\,\frac{d^2y}{dt^2}, \text{ or } \frac{x}{r}\frac{d^2x}{dt^2} + \frac{y}{r}\frac{d^2y}{dt^2};$$

and the acceleration, T, perpendicular to the radius vector is

$$\cos\theta\,\frac{d^2y}{dt^2} - \sin\theta\,\frac{d^2x}{dt^2} = \frac{x}{r}\frac{d^2y}{dt^2} - \frac{y}{r}\frac{d^2x}{dt^2}.$$

To find expressions for these accelerations in terms of r and θ, we have

$$x\,\frac{dx}{dt} + y\,\frac{dy}{dt} = r\,\frac{dr}{dt};$$

hence

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2,$$

but

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2;$$

accordingly

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+r^2\left(\frac{d\theta}{dt}\right)^2=r\frac{d^2r}{dt^2};$$

therefore

$$\frac{x}{r}\frac{d^2x}{dt^2}+\frac{y}{r}\frac{d^2y}{dt^2}=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2.$$

Also

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

Consequently, the acceleration along the radius vector is

$$P=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2. \qquad (9)$$

And that perpendicular to the radius vector is

$$T=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right). \qquad (10)$$

If the acceleration of the moving particle be always directed to the fixed point O, we have $T=0$, and hence $r^2\frac{d\theta}{dt}=$ constant; from which we infer that the radius vector describes equal areas in equal times round the point O.

29. **Areal Velocity, Areal Acceleration.**—It is obvious, geometrically, that $r^2d\theta$ represents double the area described by the line OP in the time dt; consequently $\frac{r^2d\theta}{dt}$ represents the rate of increase of double the area described by the point P round the point O. Hence $\frac{1}{2}\frac{r^2d\theta}{dt}$ is called the *areal velocity* of the point P relative to the

origin O. Similarly $\frac{1}{2}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ represents the *areal acceleration* of P relative to the same origin.

30. **Units of Time and Space.**—With respect to the units of time and space, as well as of all other quantities, it should be remarked that the units assumed must in all cases be *finite magnitudes*. For instance, the unit of time may be taken as a second, an hour, a day, or any other finite interval of time, but it should never be assumed to be an *indefinitely small portion* of time, for if so, numbers which represent finite intervals of time become infinitely great, and accordingly arguments based on such an assumption become illusory and unmeaning when applied to finite intervals of time. This remark is requisite, as fallacious proofs are sometimes given in books on dynamics from overlooking this obvious principle.

The unit of time most universally adopted is a second, as already stated. Different units of length prevail in different countries. Since in this country the foot is the standard of length, and areas and volumes are each referred to units of their own, we shall sometimes employ such units for the purpose of illustrating mechanical principles by familiar examples. But, when desirable, we shall avail ourselves of the metric system. In it the unit of length is a metre (3·2809 feet, or 39·37079 inches). From this, by the simple processes of squaring and cubing, units of area and volume are derived; and decimal multiples and submultiples are respectively indicated by the use of Greek and Latin prefixes. For example, the centimetre is the hundredth part of the length of a metre. Again, one cubic decimetre is the measure of capacity called a litre; and is about 61 cubic inches, or 1·76 pints. We shall subsequently see that a cubic centimetre of distilled water at its greatest density furnishes this system with another unit: to this the name gramme is applied. One thousand grammes are called a kilogramme, equivalent to about two and one-fifth pounds avoirdupois.

CHAPTER III.

LAWS OF MOTION.

Section I.—*Rectilinear Motion.*

31. In the preceding chapters motion has been considered from a purely *kinematical* point of view; we now proceed to consider it in connexion with the force or forces by which it is produced.

The science of Rational Dynamics is usually founded on three principles, or Laws of Motion, which have been stated in their simplest form by Newton, and are fully verified by their agreement with experience. In the present chapter it is proposed to discuss and illustrate various cases of application of these Laws, chiefly when the forces supposed to act are constant both in direction and magnitude. The discussion of motion produced by varying forces will be dealt with subsequently. We follow Newton's method, commencing with the statement of his First Law.

32. **First Law of Motion.**—*A body continues in its state of rest, or of straight uniform motion, except in so far as it is compelled to alter that state by impressed force.*

This law asserts that a body has no power or tendency in itself to alter either its velocity or the direction of its motion: this is usually called the Law of Inertia of Matter.

Hence, if a body be conceived to be set in motion, and no external force act upon it afterwards, it should continue to move indefinitely in a right line with a uniform velocity.

Conversely, if a body be in a state of uniform rectilinear motion, we infer that the forces which act on it are in equilibrium. For example, if a train be in a state of uniform motion on a horizontal railway, we infer that the force arising from the action of the steam is exactly equal, and opposite to, the entire resistance arising from friction and resistance of the air.

Hence, all questions of *uniform rectilinear motion may be* *regarded as problems of equilibrium, and treated by the principles arrived at in Statics.* In this case we suppose the body to have a motion of translation solely.

Again, if the motion of a body be not uniform, or not rectilinear, we infer that it must be acted on by some external force or forces. The connexion between the motion produced and the force which produces it is contained under the next Law.

EXAMPLE.

A railway train is moving with constant velocity along a horizontal railroad. The resistance from friction, &c., for each carriage is one-hundredth part of the pressure. Find the tension of the couplings of the last carriage, if its weight be four tons.

In this case, since the motion is uniform, the tension of the couplings must be equal to the resistance to be overcome, or to the one-hundredth part of four tons, *i.e.* $89\frac{3}{5}$ lbs.

33. **Second Law of Motion.**—*Change of motion*[*] *is proportional to the impressed motive force, and takes place in the right line in which that force is impressed.*

As this statement is very comprehensive it will be necessary to dwell on it with some detail, commencing with the case of a body under the influence of a force which acts uniformly and in the same right line during the motion. The body is supposed, in the first instance, to start from rest, and the *direction of the force to pass constantly through its centre of mass,* in which case the motion is one of translation[†] solely.

34. **Velocity Generated.**—Suppose a force to act uni-

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[*] For the present we shall consider that it is one and the same body which is acted on by forces passing through its centre of mass, in which case the force varies directly as the velocity generated in the unit of time. We shall subsequently treat of the case where the mass acted on varies also. In that case, by the word "motus," here translated motion, we must understand *quantity of motion.*

[†] A force applied at the centre of mass of a rigid body is equivalent to an indefinite number of equal and parallel forces applied to the several equal particles of which the body is conceived to be constituted; but as the forces are equal and the masses moved by each are equal, the velocities generated, in the same time, are also equal: hence the motion of the entire body is one of pure translation. The simplest case of this is that of bodies falling under the action of the force of gravity.

formly on a body, and let f denote the velocity generated at the end of the first second (taken as the unit of time), then during the next second, in accordance with our law, the uniform force will generate an additional velocity of the same amount f; and in each successive second the force generates the same additional velocity; consequently the motion is in this case *uniformly accelerated*, and the velocity at the end of t seconds (Art. 17) is given by the equation

$$v = ft.$$

Again, if the body be supposed to start with the velocity v_0 in the direction in which the force acts, we shall have for the velocity v, at the end of the time t,

$$v = v_0 + ft, \tag{1}$$

as in Art. 17.

If the force act in a direction opposite to that of the motion it is called a *retarding force*, and, if uniform, it will diminish the velocity by the quantity f during each second, and we shall have, as before, the equation

$$v = v_0 - ft.$$

The student should bear in mind that f in all cases is measured by the *velocity generated or destroyed in the moveable in each second during the motion* ; f consequently may always be regarded as an *acceleration*—a retardation being considered as a negative acceleration.

It may be observed that the entire reasoning in this Article depends on the following principle—contained in the Second Law of Motion—that the *change of velocity produced by a force in any time is independent of the previous velocity of the moveable.*

The Second Law of Motion equally applies to the case of a body acted on by any number of forces, in which case it may be stated as follows:—

If any number of forces act simultaneously on a rigid body, then, during any instant, each force produces the same change of motion in its own direction as if it had acted singly on the body.

From this it follows that forces are compounded in the same manner as velocities. The law of the composition of forces was thus established by Newton—*Leges Motus*, Cor. 2.

35. **Space Described in any Time.**—Since we have seen that in the case of a uniform force the velocity is uniformly accelerated or retarded, we can at once apply the results already arrived at in Arts. 18, 19.

Hence, the space described from rest, in the time t, is given by the formula

$$s = \tfrac{1}{2}ft^2. \qquad (2)$$

If the body start with an initial velocity v_0 along the line in which the force acts, we shall have

$$s = v_0 t \pm \tfrac{1}{2}ft^2, \qquad (3)$$

in which the upper or lower sign is taken according as the uniform force acts in the same or the opposite direction to that of the initial velocity.

It is plain that the space described in the first second from rest is $\frac{1}{2}f$, or *half the velocity acquired at the end of the second;* and, in general, the space described in any time from rest is half of that described by a body moving uniformly with the velocity acquired at the end of the time.

36. **Relation between Velocity and Space described.**—If the body start from rest, by eliminating t between the equations $v = ft$ and $s = \frac{1}{2}ft^2$, we get

$$v^2 = 2fs\,;$$

and, more generally, if v_0 be the initial velocity,

$$v^2 = v_0^2 \pm 2fs. \qquad (4)$$

From the preceding results it is seen that the question of rectilinear motion under the action of a constant force is completely solved whenever the value of the acceleration f can be determined. In a subsequent Article we shall show how this can be done in elementary cases, but before doing so we proceed to apply the preceding results to the important case of falling bodies.

37. **Vertical Motion.**—In order to get rid of the retardation caused by the resistance of the air, we shall sup-

pose the motion to take place in a vacuum. Under these circumstances it is found that all bodies, no matter what their density or chemical constitution may be, fall through the same vertical height and acquire the same velocity in the same time. That this is so is best established by means of pendulum experiments; but it can also be tested by allowing different bodies to fall in an exhausted receiver. We hence infer that the attractive force of the Earth acts equally on all bodies.

If g denote the acceleration due to the force of gravity, that is the *increment of velocity per second acquired by a body falling in a vacuum*, then, from what has been stated, the value of g is the same for all bodies at the same place on the Earth's surface.

Again, since at any place the force of gravity may be assumed as a constant force (*i. e.* within moderate distances from the Earth's surface), we may apply to the case of falling bodies the results arrived at in the preceding Articles by substituting g in place of f. Hence, if the body start from rest, we have

$$v = gt, \quad s = \tfrac{1}{2}gt^2, \quad v^2 = 2gs. \qquad (5)$$

Again, if it start downwards with a given vertical velocity v_0,

$$v = v_0 + gt, \quad s = v_0 t + \tfrac{1}{2}gt^2, \quad v^2 = v_0^2 + 2gs. \qquad (6)$$

If the body be projected *vertically upwards* with a velocity v_0, gravity becomes a uniformly retarding force, and we have

$$v = v_0 - gt, \quad s = v_0 t - \tfrac{1}{2}gt^2, \quad v^2 = v_0^2 - 2gs. \qquad (7)$$

To find in this case the height H to which the body would ascend, we make $v = 0$ in the last equation, and we get

$$H = \frac{v_0^2}{2g}. \qquad (8)$$

The time T of ascent is given in like manner by the equation

$$T = \frac{v_0}{g}. \qquad (9)$$

The subsequent motion of the body is got from equations (5), in which we suppose the body to start from rest at the height H. It immediately follows that the times of ascent and descent are equal, and that the body returns to its original position with the velocity with which it was projected upwards. For this reason we say that the *velocity* v_0 *is due to the height* H; and reciprocally, that the *height* H *is due to the velocity* v_0. We shall meet frequent applications of these expressions.

As the motion is supposed to take place in a vacuum, the preceding results can only in some cases be regarded as approximate for motion in the air.

38. **Variation of Gravity.**—It is found that the value of g varies, within small limits, from place to place on the Earth's surface. It increases with the latitude, and when referred to feet and seconds, has its least value, 32·091, at the equator, and its greatest, 32·255, at the pole. It also diminishes as the body is raised above the Earth's surface, since the attraction of the Earth varies as the inverse square of the distance from its centre. The value of g at London, referred to the same units, is 32·19, and this may be employed, in ordinary calculations, as an average value.

It will be seen subsequently that the rotation of the Earth on its axis has the effect of diminishing the velocity of a falling body; and, accordingly, the observed value of g is the difference between its value arising from the Earth's attraction and the component of the centrifugal acceleration in the vertical direction.

As a rough approximation we may assume $g = 32$; and, when numerical results are required, this may be taken as its value in these and all subsequent examples, unless otherwise specified, inasmuch as they are given chiefly for the purpose of familiarizing the student with the application of mechanical principles.

EXAMPLES.

1. Find the velocity acquired in 5 minutes by a falling body, assuming $g = 32·19$. *Ans.* 9657 feet.

2. In what time will a falling body acquire a velocity of 400 feet if it start from rest? *Ans.* 12·5 sec.

3. If a body move under the action of a constant force, its *average velocity* during any time is an arithmetical mean between its velocities at the commencement and the end of that time?

4. If one minute be taken as the unit of time, what should be taken as the value of g?

Ans. The velocity per minute acquired in one minute by a falling body, or 115,200 feet.

5. Two bodies start together from rest, and move in directions at right angles to each other. One moves uniformly with a velocity of 3 feet per second; the other moves under the action of a constant force: determine the acceleration due to this force if the bodies at the end of 4 seconds be 20 feet apart.

Ans. 2 feet per second.

6. If a uniform force generate in a body a velocity of 30 feet a second after describing 25 yards, find the acceleration. *Ans.* $f = 6$.

7. A stone is let fall from a height into a well, and is heard to strike the water after t seconds; find the depth of the well, assuming the velocity of sound to be V, and neglecting the resistance of the air. The required height h is got by solving the equation

$$\frac{h}{V} + \sqrt{\frac{2h}{g}} = t.$$

In applying this equation practically it may be observed that $\frac{h}{V}$ is, in all cases, small in comparison with t: accordingly, if we transpose and square, we get, neglecting $\frac{h^2}{V^2}$ in comparison with $\frac{2ht}{V}$,

$$h = \frac{gVt^2}{2(V + gt)},$$

8. A person drops a stone into a well, and after three seconds hears it strike the water. If the velocity of sound be 1127 feet per second, find the depth of the water. *Ans.* 132·68 feet.

9. Prove that the spaces described by a falling body in successive equal intervals of time are proportional to the series of odd numbers.

10. A body moves from rest under the action of a constant force during four seconds, when the force is supposed to cease; in the next five seconds the body describes 200 feet; find the acceleration due to the constant force—(1) if one second; (2) if one minute be taken as the unit of time.

Ans. (1) 10, (2) 36000.

11. A body is projected upwards with any velocity, and t, t' denote the times in which it is respectively above and below the middle point of its path; find the value of $\frac{t}{t'}$. *Ans.* $\sqrt{2} + 1$.

12. Assuming g to be represented by 32 when the units of space and time are one foot and one second; what number would represent its value if one mile and one day be taken as the units? *Ans.* $45242181\frac{9}{11}$.

13. A ball is dropped from the masthead of a ship sailing n miles an hour. Through how many feet must it have fallen when the direction of its motion is inclined at 45° to the horizon?

Ans. $\frac{121\,n^2}{3600}$.

vert. vel: in time $t = gt$

horiz: vel. $= \frac{n \times 1760 \times 3}{60 \times 60}$

$\therefore \frac{n \times 1760 \times 3}{60 \times 60}$ ... $s = \frac{1}{2} gt^2$

39. **Acceleration Varies as Pressure.**—If we suppose different forces to act uniformly during equal times on the same body, it follows from the Second Law of Motion that the *forces will be to one another in the same ratio as the velocities generated in equal times.*

If we suppose the time of action to be one second, the velocities generated are represented by the corresponding accelerations f and f'. Also, if F, F' denote the statical* measures of the forces, *i.e.* the pressures which they are capable of producing, we have

$$F : F' = f : f'. \tag{10}$$

If one of the constant forces be the attraction of the Earth, since its statical measure is W, or the weight of the body moved; and since g is the corresponding acceleration, we have

$$F : W = f : g; \tag{11}$$

hence

$$f = \frac{F}{W} g. \tag{12}$$

This equation enables us to determine the velocity generated in one second by a constant force at any place whenever the pressure F which measures the force is known, and also the weight of the body. We suppose, as stated already, that the body is rigid, and that the force F acts through its centre of mass. When f has been determined by the foregoing equation, and the force continues to act uniformly, we may apply the results arrived at in the preceding Articles to determine the subsequent motion (see Art. 36).

40. **Mass.**—If we attempt to move a body we find that with the same amount of exertion on our part different results are produced in different bodies.

For example, the same effort which would project a small stone a considerable distance will move a large one but slightly.

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* The magnitude of a force is estimated in Statics by the weight which it is just capable of supporting. Thus, a force which is capable of supporting a weight of 112 lbs. is called a force of 112 lbs., &c.

In this way our ordinary experience suggests to us that the amount of the acceleration produced in a body by a force depends not only on the magnitude of the force but also on the body which is moved. When exact experiments are carried out, it is found that the same force acting on different bodies produces different accelerations, and that different forces acting on the same body produce accelerations proportional to the forces. Hence we conclude that the acceleration produced in the motion of a body by a force is equal to that force multiplied by a factor which is invariable for the same body, but which varies for different bodies.

Conversely, if F denote the magnitude of a force, and f that of the acceleration thereby produced, we have the equation

$$F = mf, \tag{13}$$

where m is always the same for the same body, but varies for different bodies. This quantity m is called the Mass of the body, and is estimated, like other quantities, by comparing it with a standard quantity of the same kind. It is found that at any fixed place on the Earth's surface the weight of a body (if permitted to accelerate its motion) produces an acceleration which is the same for all bodies (Art. 37). Now W being the weight and g the acceleration thereby produced, we have, as above, $W = mg$; but g is the same for all bodies at the same place, hence W is proportional to m; or, in other words, if there be two bodies whose weights are W, W', and whose masses are m, m', we have $\frac{W}{W'} = \frac{m}{m'}$. Hence, in order to find the ratio of the masses of two bodies, we have only to find the ratio of their weights at the same place.

Examples.

1. A uniform pressure of 6 lbs. is applied in a horizontal direction to a body of 10 lbs. mass placed on a smooth horizontal table. Find—(1) the velocity generated in one second; (2) that acquired after describing 500 yards along the plane. *Ans.* (1) $19\frac{1}{5}$; (2) 240.

2. If a uniform pressure of 3 lbs. produce a velocity of 10 feet in the first second, find the weight of the body acted on. *Ans.* 9·6 lbs.

3. Find the pressure which, acting uniformly during one second, will generate in a body of one ton mass a velocity of 10 miles per hour.

Ans. 9 cwt. $18\frac{2}{3}$ lbs. pressure.

4. If a pressure of one ounce act uniformly on a body of one pound mass, find the velocity generated from rest in one minute. *Ans.* $\frac{15}{4}g$.

5. If a uniform force generate in a mass of 10 lbs. a velocity of 30 feet after describing 25 yards, find the statical measure of the force. *Ans.* $\frac{60}{g}$ lbs.

6. A pressure F, acting on a mass of weight W, generates in it a velocity V, in the time T; find the value of F. *Ans.* $F = W\dfrac{V}{gT}$.

7. A train weighing 100 tons acquires on a horizontal railroad in four minutes a velocity of 30 miles an hour; find the statical measure of the excess of the moving above the retarding pressure, each being assumed to be uniform.

Ans. 11 cwt. 1 qr. $23\frac{1}{3}$ lbs.

8. A train weighing 60 tons is impelled along a horizontal road by a constant pressure of 720 lbs. Supposing it to start from rest, find its velocity at the end of one minute—(1) neglecting friction; (2) assuming the resistance of friction, air, &c., to be 8 lbs. per ton. *Ans.* (1) $\frac{9}{28}g$; (2) $\frac{3}{28}g$.

9. If a uniform force of 6 lbs. produce in a second a velocity of 0·634 feet in a body, express the quantity of matter in the body in terms of cubic feet of water, assuming the weight of a cubic foot of water to be $62\frac{1}{2}$ lbs. and $g = 32{\cdot}19$. *Ans.* 4·87.

10. Find the magnitude of the pressure which, acting uniformly during $\frac{1}{10}$th of a second, would generate a velocity of 100 feet per second in a body of 16 lbs. mass. *Ans.* 500 lbs.

11. A mass, which weighs 450 lbs., is placed on a perfectly smooth table: a uniform horizontal pressure is exerted on it which increases its velocity 3 feet in every second; find the magnitude of the pressure in lbs.

Ans. 41 lbs., assuming $g = 32{\cdot}19$.

41. **Motion on a Smooth Inclined Plane.**—We next suppose a body, *starting from rest*, to slide under the influence of gravity down a perfectly smooth inclined plane. Let i denote the inclination of the plane to the horizon, and W the weight of the body. Resolve W into its components, $W \sin i$ acting parallel to the plane, and $W \cos i$ perpendicular to the plane. The motion down the plane is evidently due to the former component, since the latter only causes pressure on the plane.

As the force along the plane is constant and acts in the direction of motion, we get, substituting $W \sin i$ for F in (12),

$$f = g \sin i. \qquad (14)$$

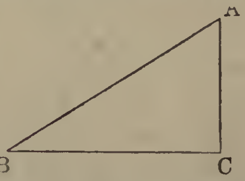
Hence, if $g \sin i$ be substituted for f in the formulæ in Arts. 35 and 36, we get

$$v = gt \sin i, \quad s = \tfrac{1}{2} gt^2 \sin i, \quad v^2 = 2gs \sin i. \qquad (15)$$

We assume that the body slides without rolling along the plane, as otherwise the motion would not be one of pure translation.

42. **Velocity acquired in Moving down an Inclined Plane.**—Let l represent AB, the length of the plane, and h its height AC; then if v be the velocity acquired on arriving at B, we have

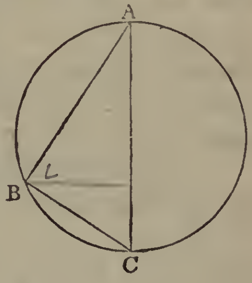
$$v^2 = 2gl \sin i = 2gh. \qquad (16)$$



Accordingly, *the velocity acquired at any point in the descent of a body down a smooth inclined plane is that due to the vertical height through which the body has descended.* This is a particular case of an important principle which shall be subsequently considered.

43. **Time of Descending a Chord of a Vertical Circle.**—We next proceed to show that the time of descent down any chord of a vertical circle, starting from its highest point, is constant.

Let AC be the vertical diameter of the circle, AB any chord drawn from A. Join BC; then, $\sin i = \sin BCA = \frac{AB}{AC}$; and, if T be the time of descent for AB, we have, by (15),



$$AB = \tfrac{1}{2} g T^2 \frac{AB}{AC}, \quad \therefore \; T^2 = 2\frac{AC}{g};$$

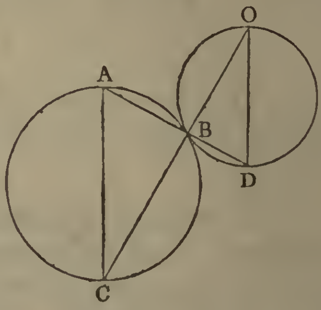
hence

$$T = 2\sqrt{\frac{a}{g}}, \qquad (17)$$

where a denotes the radius of the circle.

Hence, the time down any chord such as AB of this circle is constant. It can at once be seen, in like manner, that the time of descent down BC has the same value.

44. **Line of Quickest Descent to a Circle.**—To find the right line down which a particle under the action of gravity would descend in the shortest time from a given point O to a given vertical circle.

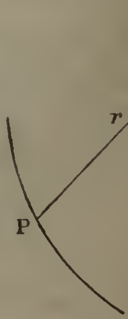


Draw AC, the vertical diameter of the circle, and join OC, meeting the circle in B, then OB is the line of quickest descent in question. For join AB, and produce it to meet the vertical drawn through O in D. Then it is obvious that the circle described on OD as diameter touches the given circle in B: consequently the time of descent down OB being the same as that down any other chord of the circle OBD, drawn from O, is less than the time down any other right line drawn from O to meet the circle ABC.

The preceding method of investigation applies equally if the point O lie inside the given circle.

45. **Line of Quickest or Slowest Descent to any Curve.**—It is easily seen from the preceding Article that the determination of the right line of quickest or slowest descent to any given vertical curve from any point in its plane reduces to the problem of drawing a circle, touching the given curve and having the given point for its highest point.

The problem admits also of being treated by the ordinary method of maxima and minima, as follows:



Suppose the curve referred to polar coordinates, the given point O being taken as pole, and the vertical OD through it as prime vector; then, if t be the time of descent down any radius vector OP, we have

$$r = \tfrac{1}{2} g t^2 \cos\theta, \quad \text{or } t = \sqrt{\frac{2r}{g\cos\theta}}.$$

Accordingly, the time t is a maximum or a minimum when $\frac{r}{\cos\theta}$ is a maximum or a minimum.

To find the maximum or minimum values, assume $u = \frac{r}{\cos\theta}$; then since $\frac{du}{d\theta} = 0$, we have

$$\cos\theta\frac{dr}{d\theta} + r\sin\theta = 0. \tag{18}$$

The solutions are obtained by combining this equation with that of the curve.

To distinguish between the maximum and minimum solutions, we proceed to differentiate the equation

$$\frac{du}{d\theta} = \frac{\cos\theta\frac{dr}{d\theta} + r\sin\theta}{\cos^2\theta};$$

observing that, in this case, $\cos\theta\frac{dr}{d\theta} + r\sin\theta = 0$. Hence (*Diff. Calc.*, Art. 138), t is a minimum or a maximum according as $r + \frac{d^2r}{d\theta^2}$ is positive or negative.

These results can be readily verified from geometrical considerations.

EXAMPLES.

1. If the hypotenuse of a right-angled triangle be placed in a vertical position, prove that the times of descending from rest will be the same for each of its sides.

2. Prove that the velocity acquired down any chord, terminated at the lowest point of a vertical circle, is proportional to the length of the chord.

3. If the length of an inclined plane be 150 yards, and its inclination 30°, what velocity would a body acquire in descending it?

Ans. 40 yards per second.

4. A body slides down a smooth inclined plane of given height; prove that the time of descent varies as the length of the plane.

5. Find the inclination of a plane, of given length l, so that the velocity acquired in moving down it shall be of a given amount V.

Ans. $\sin i = \frac{V^2}{2gl}$.

6. Given the base a of an inclined plane, find its height so that the horizontal velocity acquired by descending it may be the greatest possible.

Ans. $h = a$.

7. Find the gradient in a railway that a carriage descending the plane by its own weight may move through one quarter of a mile in the first minute; and find how far the carriage will move in the next minute; friction being neglected.

(1) $\sin i = \frac{11}{480}$; (2) $\frac{3}{4}$ of a mile.

8. A body is attached by a string to a point in a smooth inclined plane, on which it rests; if it be projected from its position of rest up the plane with a velocity just sufficient to take it to the highest point to which the string allows it to go, find the time of its motion.

Ans. $t = 2\sqrt{\frac{l}{g \sin i}}$, l being length of string.

9. A groove is cut in an inclined plane, making an angle α with the intersection of the plane and the horizon. If a heavy particle be allowed to descend the groove (supposed smooth), prove that its acceleration is $g \sin i \sin \alpha$; where i denotes the inclination of the plane.

10. If two vertical circles have a common highest point, then if any line be drawn from that point, the time of descending the portion intercepted between the circles is constant.

11. Find the right line of quickest descent from a point to a given right line lying in the same vertical plane as the point.

12. Find the right line of quickest descent from a given right line to a given vertical circle.

13. Find the lines of quickest and slowest descent between two vertical circles which lie in the same plane.

14. A parabola whose latus rectum is p is placed in a vertical plane, with its axis horizontal. Find the inclination of the normal terminated by the axis down which a particle would descend in the shortest time, and find the time of its descent.

Ans. $i = 45°$, time $= \sqrt{\frac{2p}{g}}$,

15. Find the latus rectum of a parabola, so that when it is placed in a vertical plane with its axis horizontal the least time in which a particle falls from rest down a normal from the curve to the axis may be one second.

16. Prove that the chords of quickest and slowest descent from the highest, or to the lowest, point of a vertical ellipse are at right angles to each other, and parallel to the axes of the curve.

17. Show immediately, from equation (18), that the right line of quickest descent from a given point to a given curve makes equal angles with the normal at its extremity and the vertical; and verify the result geometrically.

18. An ellipse is placed with its major axis vertical; find the semi-diameter along which a particle will descend in the shortest time possible from the circumference to the centre.

Ans. It makes with the axis major the angle $\sec^{-1}(e\sqrt{2})$, where e is the eccentricity. If $e < \frac{1}{\sqrt{2}}$ the line of quickest descent is the axis major.

19. An ellipse is placed with its major axis vertical; find the line of quickest descent from the upper focus to the curve.

Ans. It makes with the axis major the angle $\cos^{-1}\frac{1}{2e}$. If $e < \frac{1}{2}$ the axis major is the required line.

20. It is required to draw to a given right line from a given point a line such that if a heavy particle move along it, starting with a given velocity, the time of reaching the given line may be the least possible. Find an equation for determining the inclination of the required line.

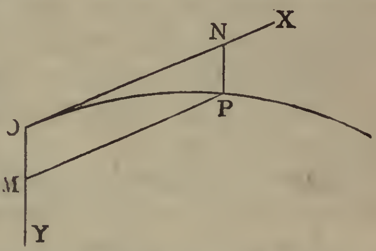
21. AB is a quadrant of a circle whose centre is O, the radius OB being horizontal; C is a point on the quadrant, and the angle $BOC = \theta$. Show that the time of falling from A to C is to that of falling from C to B as $\sqrt{\cos\frac{\theta}{2}}$ to $\sqrt{\sin\frac{\theta}{2}}$.

SECTION II.—*Parabolic Motion.*

46. **Path of a Projectile.**—We have hitherto considered rectilinear motion; we now proceed to the case of a particle projected in any direction, and acted on only by the force of gravity, which is supposed to be uniform.

In this case it is easily shown that *the path* described by the projectile is a parabola.*

For, suppose a body projected from O with the velocity V, in the direction OX, and draw OY vertically downwards.



Let ON be the space which the body, moving with the velocity V, would describe in t seconds; then, if no force were to act on the body, N would represent its position at the end of that time.

Again, as the force of gravity acts in the direction OY, it will produce its effect in that direction, by the Second Law of Motion, independently of the previous velocity of the body: *i.e.* it will produce the same effect as if the body fell freely from rest. Measure off, accordingly, $OM = \frac{1}{2}gt^2$, then OM represents the space moved through in the vertical direction in the time t.

Complete the parallelogram $OMPN$, and by the combined effect of the two motions P will be the position of the projectile at the end of the time t.

Let (x, y) be the co-ordinates of P referred to the axes OX and OY, and we have

$$x = ON = Vt, \quad y = OM = \tfrac{1}{2}gt^2.$$

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* As before, by the path described by a body we understand the path described by its *centre of mass*.

If t be eliminated between these equations, the equation of the path described is

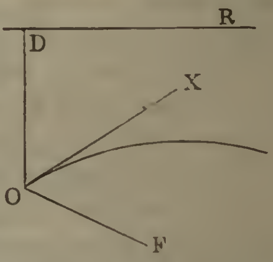
$$x^2 = \frac{2V^2}{g} y. \qquad (1)$$

This equation represents a parabola, touching OX and having its axis vertical.

If H be the height due to the velocity V (Art. 37), the equation of the parabola becomes

$$x^2 = 4Hy. \qquad (2)$$

47. **Construction for Focus and Directrix.**—From the preceding equation it follows (Salmon's *Conic Sections*, Art. 214) that H is the distance of O from the focus of the parabola, and also from its directrix.



Hence, if OD be measured vertically upwards equal to H, and DR drawn in a horizontal direction, the line DR will be the directrix of the parabolic trajectory.

Also if OF be drawn through O, making the angle $XOF = \angle XOD$, and if we take $OF = OD$; then F will be the focus of the trajectory.

Hence, as the focus and directrix of the curve are known, it is completely determined.

Again, the *velocity at any point in the trajectory is equal to that which the body would acquire in falling from the directrix.*

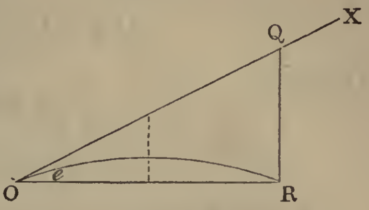
We have seen that this property holds for the point of projection: moreover, after passing through any point the body will move in the same path as if it had been projected from that point, in the direction and with the velocity that it has at the instant; therefore the property holds for any point in the path.

Hence, whenever the velocity at any point is given, the position of the directrix is completely determined.

Definition.—The angle which the direction of projection makes with the horizontal is called the *angle of elevation* of the projectile.

48. **Horizontal Range and Time of Flight.**—Let R be the point in which the projectile strikes the horizontal

plane through O; then OR is called the horizontal range, and the time T of describing the corresponding path is called the time of flight.



Through R draw RQ in the vertical direction.

Let $OR = R$, $\angle QOR = e$; then we have

$$OQ = VT, \quad QR = \tfrac{1}{2}gT^2.$$

But $QR = OQ \sin e$; hence we get

$$T = \frac{2V \sin e}{g}. \qquad (3)$$

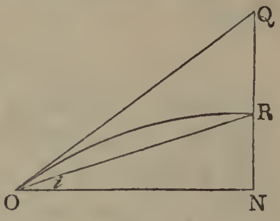
Also $R = OQ \cos e = VT \cos e = 2\,\dfrac{V^2}{g} \sin e \cos e$;

therefore
$$R = 2H \sin 2e. \qquad (4)$$

If V be given, the horizontal range is the greatest when $\sin 2e = 1$, or $e = 45°$.

The maximum horizontal range is accordingly $2H$ or double the height due to the velocity of projection.

49. **Range and Time of Flight for an Oblique Plane.**—First suppose it an ascending plane, and let i be its inclination, and e the angle of elevation QON. Then, as before, we have



$$OQ = VT, \quad QR = \tfrac{1}{2}gT^2.$$

But in the triangle QOR, we have

$$\frac{QR}{OQ} = \frac{\sin(e-i)}{\cos i};$$

hence

$$\frac{\sin(e-i)}{\cos i} = \frac{gT}{2V},$$

or

$$T = \frac{2V}{g}\,\frac{\sin(e-i)}{\cos i}. \qquad (5)$$

Also the range

$$OR = OQ\,\frac{\cos e}{\cos i} = VT\,\frac{\cos e}{\cos i};$$

therefore

$$R = \frac{2V^2}{g}\,\frac{\sin(e-i)\cos e}{\cos^2 i}. \qquad (6)$$

In the case of a descending plane, it is easily seen that the range and time of flight are obtained by changing the sign of i in the preceding results.

For given values of V and i, R becomes a maximum when $\sin(e-i)\cos e$ is a maximum, or when

$$\sin(2e-i) - \sin i \text{ is a maximum};$$

but this is greatest when

$$2e - i = 90^\circ, \quad \text{or} \quad e = \tfrac{1}{2}(90^\circ + i).$$

Hence, the direction of elevation for a maximum range bisects the angle between the vertical and the inclined plane.

Again, since in this case $OR = RQ$, the maximum range and the corresponding time of flight are connected by the relation

$$R = \tfrac{1}{2}gT^2.$$

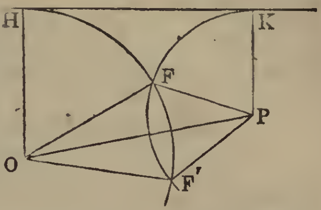
From the value of e found above it follows immediately that the focus of the parabola, in this case, lies on the inclined plane.

50. *Given the Velocity of Projection to find the elevation in order to strike a given object.*—Here, in formula (6), we are given V, R, and i, to find e. Hence, $\sin(e-i)\cos e$ is given, and therefore $\sin(2e-i)$ is given, from which e can be determined.

The problem admits of a simple geometrical investigation also, as follows:—

Let O be the point of projection, and P the position of the given object. Then since the velocity of projection is given, the position of the directrix HK is known.

Hence, with O and P as centres, describe circles touching the directrix, and let F, F' be their points of intersection. These points are obviously the foci of the two parabolic trajectories which satisfy the proposed conditions. Hence the problem admits in general of two solutions.

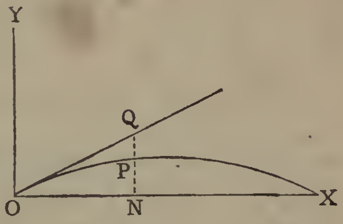


The corresponding directions of projection are found by bisecting the angles FOH and $F'OH$, as is obvious from the elementary properties of the parabola.

The problem becomes impossible when the circles do not intersect.

The range in the direction OP is obviously a maximum when the circles touch one another. In this case there is but one solution, and the focus of the parabola lies in the line OP, as already seen.

51. **Trajectory referred to Vertical and Horizontal Axes.**—Suppose OX and OY to be the horizontal and vertical lines drawn through the point of projection O, and let x, y be the coordinates of P, the position of the projectile at the end of any time t.



Let OQ be the direction of projection, and resolve the initial velocity V into its horizontal component, $V\cos e$, and its vertical, $V\sin e$. Then, since the force of gravity has no effect on the horizontal motion, the component $V\cos e$ remains constant during the motion; consequently we have

$$x = ON = Vt\cos e.$$

Also, for the motion in the vertical direction, we get (Art. 37),

$$y = Vt\sin e - \tfrac{1}{2}gt^2;$$

therefore

$$y = x\tan e - \frac{gx^2}{2V^2\cos^2 e}$$

$$= x\tan e - \frac{x^2}{4H\cos^2 e}. \qquad (7)$$

This equation represents a parabola, whose axis is vertical, as already seen.

Again, if v be the velocity at the point P, and ϕ the angle the direction of motion makes with the axis of x, we have

$$v\cos\phi = V\cos e, \quad \text{and} \quad v\sin\phi = V\sin e - gt;$$

hence,

$$\begin{aligned} v^2 &= V^2\cos^2 e + (V\sin e - gt)^2, \\ &= V^2 - 2g\,(Vt\sin e - \tfrac{1}{2}gt^2) \\ &= V^2 - 2gy = 2g(H - y). \end{aligned}$$

Hence, as already shown otherwise, the velocity at any point is that acquired by a body falling from the directrix.

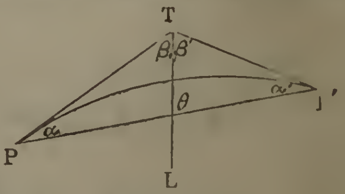
52. **Height of Ascent.**—Since vertical and horizontal motions may be considered separately, it follows that the greatest height above the horizontal plane is that to which a body projected vertically with the velocity $V\sin e$ would ascend. This (by Art. 37), is

$$\frac{V^2\sin^2 e}{2g}, \quad \text{or } H\sin^2 e.$$

Also, the time of ascent is $\dfrac{V\sin e}{g}$, from the same Article: a result which can also be obtained by finding the maximum value of y in equation (7). From these the same expressions as before for the range and the time of flight can be easily deduced: for, the whole time of flight is obviously double that of reaching the highest point; and the range is got by multiplying the value so found by $V\cos e$.

53. *If PT, $P'T$ be the tangents at two points P, P' on a parabolic trajectory, and v, v' the corresponding velocities, to prove that*

$$v : v' = PT : P'T. \qquad (8)$$



For, the line joining T to the middle point of PP' is vertical, being parallel to the axis of the parabola. Again, let

$$\alpha = \angle TPP', \quad \alpha' = \angle TP'P, \quad \beta = \angle PTL, \quad \beta' = \angle P'TL.$$

Then, since the horizontal component of the velocity at P is equal to that at P', we have

$$v \sin\beta = v' \sin\beta',$$

or

$$\frac{v}{v'} = \frac{\sin\beta'}{\sin\beta} = \frac{PT}{P'T}.$$

Also, since

$$\frac{PT}{P'T} = \frac{\sin\alpha'}{\sin\alpha},$$

we get

$$v \sin\alpha = v' \sin\alpha'. \qquad (9)$$

54. **Lemma.**—If θ be the angle BDC which a right line CD drawn from the vertex makes with the base of a triangle ABC, we have

$$AB \cot\theta = BD \cot A - AD \cot B. \qquad (10)$$

For, draw CN perpendicular to AB, and we have

$$\cot A - \cot\theta = \frac{AD}{CN}, \text{ and } \cot\theta + \cot B = \frac{BD}{CN};$$

therefore

$$AD(\cot\theta + \cot B) = BD(\cot A - \cot\theta),$$

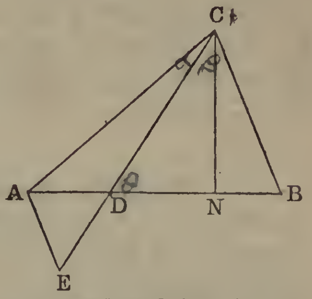
or

$$AB \,.\, \cot\theta = BD \cot A - AD \cot B.$$

Again, if α and β be the angles which CD makes with AC and BC respectively, we have

$$AB \cot\theta = AD \cot\alpha - BD \cot\beta.$$

This follows at once by drawing AE parallel to BC, and applying the preceding result.



55. *Being given the direction and the velocity of projection, to find the velocity with which a projectile would strike an oblique plane, and also the direction of its motion at the instant of impact.*

Let i be the inclination of the plane to the horizon, then, by the preceding lemma (see figure on last page),

$$\cot\alpha - \cot\alpha' = 2\tan i. \qquad (11)$$

Hence, the angle α' is determined from the known angles α and i.

Again, since $v \sin \alpha = v' \sin \alpha'$, we have

$$v' = \frac{v \sin \alpha}{\sin \alpha'},$$

which determines v'.

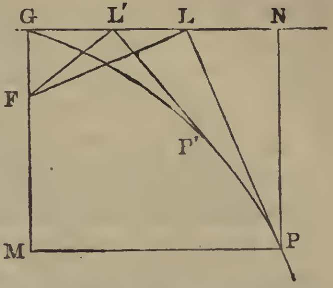
If the projectile impinge at right angles on the plane, we have $\alpha' = 90°$; therefore $\cot \alpha = 2 \tan i$, which determines α, or the corresponding angle of elevation. Also the velocity with which the projectile strikes the plane is $v \sin \alpha$ in this case.

56. **Motion on a Smooth Inclined Plane.**—In our discussion of motion on an inclined plane in Art. 41 the moveable was supposed to start from rest: in this case the motion is rectilinear. It is also rectilinear if the initial motion has place in the direction of the line of greatest slope in the plane. But when the body is projected along the plane in any other direction the problem is the same as that previously discussed, namely, the motion of a projectile acted on by a constant force, parallel to a given direction. Its path along the plane is, accordingly, a parabola; and its axis is in the direction of the line of greatest slope.

57. **Morin's Apparatus.**—We conclude with a short description of the apparatus, designed by Poncelet, and constructed by Morin, for experimentally exhibiting the laws of falling bodies.

A cylinder is made by clock-work mechanism to revolve around a fixed vertical axis. A weight is suspended at the summit of the cylinder close to the outer surface and between two vertical guides. When the rotation has become perfectly uniform, the weight is allowed to fall. A pencil, attached to the falling weight, is so arranged as to trace a line on a sheet of paper, which is wrapped tightly around the revolving cylinder. When the paper is taken off and unrolled on a plane surface, the curve traced on it by the pencil is found to be a parabola.

That this curve is a parabola may be shown in the following manner:—



Let $GP'P$ represent the curve traced out by the pencil. Draw the tangent GL to the curve at the initial point G, and at any point P draw the tangent PL, and erect LF perpendicular to it at the point L. Make a corresponding construction for the other points on the path; then the lines LF, $L'F$, &c., are all found to intersect in a common point F. This is a characteristic property of the parabola which has its focus at F, and its vertex at G. Having found the curve to be a parabola, we can show that the motion of the weight has been uniformly accelerated. Let PM, PN be the coordinates of P, referred to the axes GL, GF, then if t denote the time in which the moving weight arrived at the position P, the line PM will be equal to the arc of the circle through which a point on the circumference of the cylinder has rotated in the time t. Let V denote the constant velocity of any point on the circumference of the cylinder, and we get $PM = Vt$.

Again, from the property of the parabola,

$$PM^2 = 4FG \times MG.$$

Accordingly

$$MG = \frac{PM^2}{4FG} = \frac{V^2}{4FG}t^2;$$

but MG is the space through which the weight has descended vertically in the time t; hence the spaces described by the falling body vary as the squares of the times; its motion consequently is uniformly accelerated.

Comparing with the equation

$$s = \tfrac{1}{2}gt^2,$$

we get

$$g = \frac{V^2}{2.FG};$$

that is, the distance of the focus of the parabola from its summit is equal to the height due to the velocity of a point on the surface of the rotating cylinder.

The student can easily prove that the parabola described is the same as that of a body projected horizontally from a point with the velocity V.

58. In the preceding investigations we have neglected the effects of the resistance of the air. When this is taken into account the problem becomes one of great uncertainty, arising from the law of resistance of fluids not being accurately known, and from the difficulties still remaining in integration of the equations of motion, when the law of resistance is assumed. The most generally received theory is that the resistance of fluids is proportional to the square of the relative velocity of the fluid and the moveable. When the resistance of the air is taken into account, it is easily shown that the preceding results are not even approximate, especially in cases of high velocity; such, for instance, as shot and shell projected by artillery.

EXAMPLES.

1. Determine the elevation of a projectile, so that its horizontal range may be equal to the space to be fallen through to acquire the velocity of projection.
Ans. $e = 15°$.

2. If a number of particles be projected simultaneously from the same point with a common velocity, but in different directions, prove that at any subsequent instant they will all be situated on the surface of a sphere.

3. Given the horizontal range and the time of flight of a projectile; find its initial velocity and angle of elevation.

4. If a body be projected obliquely on a smooth inclined plane, the path in which it moves will be a parabola? Find the position of the focus and directrix of the parabola when the initial velocity and direction of motion are given.

5. Given the velocity with which a shot is projected from a certain point; find the locus of the extremities of the maximum ranges on inclined planes passing through that point.

6. If a body be projected with a velocity of 100 feet per second from a height of 66 feet above the ground, in a direction making an angle of 30° with the horizon: find when and where it will strike the ground.
Ans. Time = $4\frac{1}{8}$ sec. Range = 357·23 feet.

7. If A, B be two points on a parabolic trajectory, prove that the time of passage from one to the other is proportional to $\tan\phi - \tan\phi'$; where ϕ, ϕ' represent the inclinations to the horizon of the tangents drawn at A and B.

8. Given the initial velocity, find the angle of elevation, that a projectile should just clear a wall at a given distance from the point of projection. Find also the distance at which the body strikes the ground afterwards.

9. A piece of ordnance, under proof at Woolwich, at a distance of 50 yards from a wall 14 feet high, burst, and a fragment of it originally in contact with the ground, after just grazing the wall, fell 6 feet beyond it on the opposite side. Find how high it rose in the air.

10. When the velocity of projection is given, all the parabolas which can be described in the same plane by a projectile are enveloped by a fixed parabola: prove this, and hence find the maximum range on a given plane.

11. A body is projected with a velocity of 100 feet, in a direction inclined at an angle of 60° to the horizon: find its least velocity during the motion, and the time of attaining it. *Ans.* 50 feet, 2·7 seconds.

12. If two bodies be projected simultaneously, with a common velocity, from the same point on an oblique plane, one upwards and the other downwards, and if the directions of their projection make equal angles with the inclined plane: show that the times of flight are equal. The motion is supposed to take place in a plane perpendicular to the inclined plane.

13. With what velocity should a projectile be discharged at an elevation of 30°, so as to strike an object at a distance of 2500 feet on an ascent of 1 in 40?

14. Find the latus rectum of the parabola described by a projectile. The velocity at the *highest* point of the path is $V \cos e$, but it is also equal to the velocity acquired in falling from the directrix (47); therefore the latus rectum is $\frac{2V^2}{g} \cos^2 e$.

15. If a body be projected from the point A in the direction AC, and from any point C in the line a vertical line CD be drawn, meeting the curve described by the projectile in D; again, if B, the middle point of AC, be joined to D, show that BD will be the direction of the motion at D, and that the velocity at D will be to that at A as BD is to AB.

16. A number of bodies slide from rest down the chords of a vertical circle, starting from its highest point, and afterwards move freely: prove that the locus of the foci of their paths is a circle whose radius is half that of the given circle.

17. If bodies be projected from the same point with velocities proportional to the sines of their elevations, find the locus of points arrived at in a given time.

18. Two bodies are projected simultaneously from the same point with equal velocities, but in different directions: prove that the line which connects their positions at each instant moves parallel to a given line.

19. Two particles are projected from a point with equal velocities, their directions of projection being in the same vertical plane—t, t' being the times taken by the particles to reach their other common point, and T, T' the times of reaching their highest points. Show that $tT + t'T'$ is independent of the directions of projection.—*Camb. Trip.*, 1876.

20. If two particles be describing the same parabolic trajectory, prove that the right line connecting them envelopes an equal parabola.—*Ibid.*

21. A train is moving at the rate of 60 miles an hour when a ball is dropped from the roof inside one of the carriages. Prove that the ball describes a parabola in space, and find the position of the axis and directrix.

If the height of the carriage be 9 feet, and the ball rebound from the floor without loss of velocity, describe by means of a figure, drawn as nearly as you can to scale, the path of the ball in space so long as the motion continues.—*Ind. Eng. Exam.*, 1876.

22. Find the direction in which a rifle must be pointed in order that the bullet may strike a body let fall from a balloon at the instant of firing. Find also the point at which the bullet meets the body, supposing the balloon to be 220 yards high, the angle of elevation from the position of the rifleman 30°, and the velocity of the bullet two miles per minute.—*Ind. Civ. Serv.*, 1878.

SECTION III.—*Friction.*

59. **Laws of Dynamical Friction.**—Before completing our discussion of motion under the action of a constant force, it is desirable to make a few observations on the resistance, arising from friction, which takes place when one body slides on another. We shall consider only the case of motion along a fixed plane, and shall assume that the roughness of the plane is the same throughout. Under these circumstances the laws of friction—as established by experiment—may be stated as follows.

(1). The resistance caused by friction against the motion of a body sliding on a uniformly rough plane is proportional to the normal pressure which the body exerts against the plane.

(2). It is independent of the amount of surface in contact.

(3). It is independent of the velocity of motion.

(4). The ratio of the friction, during the motion, to the normal pressure is called the *coefficient* of Dynamical friction.

(5). The friction between two substances in motion is in general less than the friction in the state bordering on motion, or the Statical friction.

(6). The mutual friction varies with the nature of the surfaces in contact, and can be much diminished in amount by the use of unguents, as also by polishing the surfaces in contact.

The student will observe that the laws of Dynamical friction are in every respect similar to those of Statical friction (Minchin's *Statics*, Arts. 34–36).

For fuller information on the laws of Friction the student is referred to Jellett's *Theory of Friction.*

60. **Motion on a Rough Horizontal Plane.**—Let W be the weight of a body sliding on a uniformly rough horizontal plane, and μ the relative coefficient of friction; then,

as in this case the normal pressure is represented by W, the friction is μW; and since it acts as a retarding force we get by Art. 39,

$$f = \frac{\mu W}{W} g = \mu g. \tag{1}$$

Accordingly, substituting $-\mu g$ for f in the equations of Arts. 34, 35, and 36, we get

$$\left.\begin{aligned} v &= V - \mu g t \\ v^2 &= V^2 - 2\mu g s \\ s &= Vt - \tfrac{1}{2}\mu g t^2 \end{aligned}\right\} \tag{2}$$

By means of these equations the motion is completely determined whenever μ, the coefficient of friction, and V, the initial velocity, are known.

To find when the body is brought to rest by the friction, we make $v = 0$ in the first of these equations, and the required number of seconds is $\frac{V}{\mu g}$. Again, the space moved over before the body is brought to rest is given by $2\mu g s = V^2$.

61. **Motion on a Rough Inclined Plane.**—Suppose a body of weight W to slide on a uniformly rough plane, of inclination i; then resolving W into its components, $W \cos i$ and $W \sin i$, the former, $W \cos i$, represents the pressure on the plane; and accordingly the friction is represented by $\mu W \cos i$; and since it acts against the motion, we have for the total force producing motion down the plane the expression $W \sin i - \mu W \cos i$. If this value be substituted for F in equation (12), Art. 39, we get

$$f = g(\sin i - \mu \cos i). \tag{3}$$

If ϕ be the limiting angle of resistance for the plane, *i.e.* if $\mu = \tan \phi$, the preceding formula becomes

$$f = g\frac{\sin(i - \phi)}{\cos \phi},$$

for a body sliding down the plane.

The corresponding equations connecting velocity, time, and space, are had by substituting this value for f in the formulae of Arts. 35 and 36.

If the body be projected up the plane, in a direction at right angles to the intersection of the plane with the horizon, the retarding force is represented by $W \sin i + \mu W \cos i$: hence the value of f becomes

$$f = -g(\sin i + \mu \cos i) = -g\,\frac{\sin(i+\phi)}{\cos \phi}, \qquad (4)$$

when we introduce for μ its value $\tan \phi$. The equations connecting s, v, t can be found immediately, as before.

EXAMPLES.

1. A body is projected with a velocity of 100 feet per second along a rough horizontal plane: find, assuming $\mu = \frac{1}{12}$, (1) the time in which it is brought to rest by friction; (2) the whole space passed over.
Ans. (1) $37\frac{1}{2}$ seconds; (2) 625 yards.

2. A body is projected with a velocity of 100 yards per minute along a rough horizontal plane, and is brought to rest in 10 seconds: find the coefficient of friction. *Ans.* $\mu = \frac{1}{64}$.

3. A train, of 10 tons weight, is impelled by steam along a horizontal railroad with a constant pressure of 630 lbs. If the friction be 7 lbs. per ton, calculate—(1) the velocity, in miles per hour, after moving from rest for one minute; and (2) the space described in that time; neglecting the resistance of the air, &c.
Ans. (1) $32\frac{8}{11}$; (2) 480 yards.

(*b*) If the steam be shut off, find how far the train would run before it is brought to rest by friction. 2 miles 320 yards.

4. A body projected with a velocity of 30 feet is brought to rest after sliding 100 yards on a rough horizontal plane; find the coefficient of friction.
Ans. $\frac{3}{64}$.

5. A body is projected up a plane, of 20 yards length and 30° inclination, with a velocity of 50 feet per second: find the coefficient of friction that the body should just arrive at the top of the plane.
Ans. $\mu = \dfrac{29}{96\sqrt{3}}$.

6. Two masses, M, M', connected by a string, slide down a rough inclined plane in a vertical plane at right angles to the intersection of the former with the horizon: if the coefficients of friction be μ and μ', respectively, prove that the acceleration down the plane is $g\left(\sin i - \dfrac{M\mu + M'\mu'}{M + M'} \cos i\right)$.

7. A body slides down a rough roof and afterwards falls to the ground: find the whole time of motion.

8. Several bodies start from the same point and slide down different inclined planes of the same roughness: find the locus of their positions after the lapse of a given time. Find also the locus of the positions arrived at with a common velocity.

9. A rough plane makes an angle of 45° with the horizon; a groove is cut in the plane making an angle α with the intersection of this plane and the horizontal plane; if a heavy particle be allowed to descend the groove from a given height h, find the velocity with which it reaches the horizontal plane.

Ans. $\sqrt{\dfrac{2gh(\sin\alpha - \mu)}{\sin\alpha}}$.

10. A body moves from rest down an inclined plane whose inclination is 30°, and limiting angle of resistance 15°; find the velocity acquired if the length of the plane be 200 feet.

Here $v^2 = 400\ g \tan 15°$; therefore $v = 58{\cdot}56$ feet.

11. A railway train is moving up an incline of 1 in 120 with a uniform velocity. Find the tension of the couplings of the carriage which is attached to the engine; assuming the weight of the train (exclusive of the engine) to be 80 tons, and the friction 8 lbs. per ton.

Ans. 19 cwt. $5\frac{1}{3}$ lbs.

12. In the same case, if the acceleration of the train be 2 feet per second, find the tension of the couplings.

Here we must add to the preceding $W\dfrac{f}{g}$, i. e. 5 tons; and the entire tension is nearly 6 tons.

SECTION IV.—*Momentum.*

62. **Force measured by Quantity of Motion generated in Unit of Time.**—The quantity of matter in a body is called its mass. In order to compare two bodies as to the amount of matter contained in each, it is necessary to consider what is meant when we state that two bodies have equal masses. Accordingly, we define two masses to be equal when equal forces applied to each produce equal velocities in the same time. Let f as before denote the velocity generated in one second by a uniform force F applied at the centre of gravity of a rigid body; then an equal force F applied to a second body of the same mass will generate in it likewise the same velocity f in one second. If now the two bodies be conceived united, we see that it takes double the force F to generate the velocity f in the body so united. By repeating this process any number of times, f remaining the same, we conclude that the amount of force requisite to generate a velocity f in a body

varies in the same proportion as the quantity of matter or mass of the body.

Again, if the mass remain the same, F varies as the acceleration f (Art. 39). Hence, if the mass and the acceleration both vary, the force must vary as their product; *i. e.* as mf.

The product of its mass and the velocity which a body has at any instant is called its *quantity of motion* or *momentum* at that instant. Accordingly we conclude that F varies as the *quantity of motion it can generate in one second* (taken as the unit of time), the force being supposed to act uniformly during that time. Consequently, by properly assigning the units of force and of mass, we may write this result in the form*

$$F = mf. \qquad (1)$$

Since the velocity (g) which gravity can produce in one second is the same for all bodies, the quantity of motion gravity can generate in one second in a falling body of mass m is represented by mg; hence equation (1) assumes in this case the form

$$W = mg; \qquad (2)$$

in which the units of mass and weight are connected in such a manner that when one is fixed the other is also determined.

63. **Absolute Unit of Force.**—From equation (1) the unit of force is defined as the *force which acting uniformly during the unit of time on a unit of mass produces a unit of velocity.* This is called by Gauss the *absolute unit of force.*

The most convenient *unit of mass* in the British Isles is the mass contained in one *standard pound* avoirdupois.

Hence, adopting as before a second as the unit of time, and a foot as the unit of length, the absolute unit of force is that which acting during one second would produce in a standard pound mass a velocity of one foot per second. This unit of force is sometimes called a *poundal.* Hence, if $g = 32{\cdot}19$, with reference to the preceding units the *unit of force* is $\frac{1}{32{\cdot}19}$ part of the attraction of the earth, at London, on a standard pound; *i.e.* about half an ounce, approximately.

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* This equation has been already arrived at in Article 40 from other considerations.

In the metric system the force which in one second would generate a velocity of one centimetre per second in a gramme of matter is called a *dyne*. Hence, since 1 lb. = 453·6 grammes, and 1 foot = 30·48 centimetres, one poundal is approximately 13825 dynes.

64. **Gravitation Units of Force and Mass.**—In practical questions concerning bodies on the earth's surface, it is in general more convenient to measure forces by weights, and to speak of a force of so many pounds weight. In this system the unit of force is the weight at some definite place (London) of the pound mass, or of a kilogramme when the metric system is taken. This is called the gravitation or statical measure of force; and since the unit of force in this system, acting on one pound mass for one second, produces a velocity of 32·19 feet per second, we see that this unit is 32·19 times the absolute unit. Moreover, since the weight of a body varies, within certain small limits from place to place (Art. 37), when scientific accuracy is required we must correct for the change in the value of g due to any difference in altitude or latitude from those of the place to which the standard was originally referred.

In practice this correction seldom requires to be taken into account, as the variation in the value of g is generally too small to affect the result appreciably (Art. 38).

EXAMPLES.

1. An ounce being taken as the unit of mass, a second as the unit of time, and an inch as the unit of length, compare the unit of force with the weight of one pound.

Here the unit of force is that which in one second would generate a velocity of one inch per second in an ounce mass; and therefore is $\frac{1}{12 \times 16 \times 32{\cdot}2}$ part of the weight of one pound, or 1·25 grains.

2. Determine the unit of time in order that g may be expressed by unity when the foot is the unit of length.

Ans. $\frac{1}{8}\sqrt{2}$ seconds.

3. Find the units of space and time in order that the acceleration of a body falling in vacuo and the velocity it acquires in one minute may respectively be the units of acceleration and of velocity.

65. **Two Classes of Forces.**—There are two classes of forces to be considered in Dynamics: one, such as gravity and

those hitherto discussed, which require a finite time to produce a finite change of velocity. Forces of this class, when uniform, are, as has been stated, measured by the change produced in one second (taken as unit of time) in the momentum of the body acted on. There is another class, called ordinarily *impulses*, such as *blows*, sudden *impacts*, &c., which act only during a very *short* time, but are capable of producing a finite change of velocity in that time.

These are sometimes called *instantaneous* forces; it is necessary, however, to observe that force in all cases requires some time to produce its effects, though that time may be exceedingly small. In fact, we cannot conceive that a force could produce any change in the velocity of a body if its time of action were absolutely nothing.

Forces of the former class are frequently styled *finite* or continuous forces, to distinguish them from the other class, namely, *impulsive* forces.

It should be observed that whenever both *impulsive* and *finite* forces act at the same time on a body, the latter may in general be neglected in determining the motion at the instant; since the effects produced by them, in the time during which the impulsive forces act, are so small that they may be neglected in comparison with the effects of the impulses.

66. **Impulses.**—*The measure of an impulse*, i.e. of a force of great intensity, which acts during a very short time, and then ceases, is the *whole change in the quantity of motion which it communicates to the body on which it acts.*

Accordingly, the proper measure of all forces in Dynamics is a quantity of motion produced. The two classes of forces, considered above, must be considered as differing in degree but not in kind: an impulse being regarded as a force of very great intensity, acting during a very short time.

EXAMPLES.

1. If a ball discharged vertically from a gun ascends to a height h, to what height ought it to ascend if the charge of powder be doubled, neglecting the resistance of the air? *Ans.* $4h$.

2. If a shot weighing a ounces with a charge of p ounces of powder rises to a height of h feet, to what height ought a shot of b ounces, with a charge of q ounces, to ascend from the same gun? *Ans.* $\left(\frac{a}{b}\right)^2\left(\frac{q}{p}\right)^2 h$.

67. General Equations of Motion of a Particle.—Suppose that the force F acts as before in the line of motion of the mass acted on, but that it varies continuously, then we may consider that in an indefinitely small portion of time its intensity is unaltered. The variable acceleration f, caused by it, is determined by the equation $F = mf$: hence, as in Art. 21, we have at any instant

$$F = mf = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}. \qquad (3)$$

Hitherto the motion has been supposed rectilinear. In the case of curvilinear motion the last equation expresses the tangential component of the force, and it can be similarly seen (Art. 25) that the normal component is expressed by $m\frac{v^2}{\rho}$. We now proceed to consider the motion of a particle of mass m, under the action of any forces. If the particle be referred to a system of rectangular axes in space: and x, y, z be the coordinates of its position at any instant, *i. e.* at the end of the time t, reckoned from any fixed instant, the components of its velocity parallel to the axes of coordinates are (Art. 12) represented by $\dot{x}$, $\dot{y}$, $\dot{z}$.

Resolve the whole force acting on the particle at the instant into three components, parallel to the axes of x, y, z, respectively; and let these components be represented by X, Y, Z; then, since by the Second Law of Motion each of these forces produces its change of velocity in its own direction, we deduce from what precedes (see Art. 24) the equations

$$X = m\frac{d}{dt}\left(\frac{dx}{dt}\right) = m\frac{d^2x}{dt^2}. \qquad (4)$$

$$Y = m\frac{d^2y}{dt^2}, \quad Z = m\frac{d^2z}{dt^2}.$$

These are called the differential equations of motion of the particle; and the solution of the problem depends in each case on the integration of these equations.

As already stated, the preceding equations hold for the motion of any rigid body, provided the direction of the force which acts on it always passes through its centre of gravity.

SECTION V.—*Action and Reaction.*

68. **Third Law of Motion.**—*Reaction is always equal and opposite to action: that is, the mutual actions of two bodies are always equal and take place in opposite directions.*

On this law Newton remarks as follows:—"If any person press a stone with his finger, his finger is pressed by the stone. If a horse draw a body by means of a rope, the horse also is drawn (so to speak) towards the body; for the rope being strained equally in both directions draws the horse towards the body as well as the body towards the horse, and impedes the progress of one as much as it promotes that of the other. Again, if any body impinge on another, whatever quantity of motion it communicates to that other it loses itself (on account of the equality of the mutual pressure)."

Newton verified this law experimentally in the case of the collision of spherical bodies.—See Scholium, Axiomata.

He also showed that the law holds in the case of the attraction of bodies, as follows:—

Let A, B be two mutually attracting bodies, and conceive some obstacle interposed by which their approach to one another is prevented. If the body A be acted on towards B by a greater force than B is acted on towards A, then the obstacle will be more urged by the pressure of A than by the pressure of B. The stronger pressure should prevail, and cause the system consisting of the two bodies and the obstacle to move in directum towards B; and as the force is uniform the motion would be accelerated ad infinitum, which is absurd and contrary to the first law of motion; for, by that law, such a system, as it is not acted on by any *external* force, should continue in a state of rest or of uniform rectilinear motion.

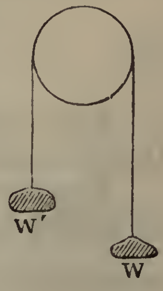
69. **Forces of Inertia.**—The fact is that force is always exhibited as a mutual action between two bodies; and this phenomenon, regarded as a whole, is described by the term *stress*, of which action and reaction are but different aspects. Thus to the action of a force producing an acceleration of motion in a body corresponds an equal and opposite reaction against acceleration; this is called the *force of inertia* of the body. It thus follows that the force of inertia of any material particle must be equal and opposite to the resultant of all the

forces which act on the particle, whether arising from the action of the other parts of the system or from that of forces external to the system. Hence in the motion of any material system, since the actions and reactions of its different parts equilibrate in pairs, we infer that there is equilibrium between the external forces which act on the system and the several forces of inertia of the different particles of which the system is composed. This is equivalent to the celebrated principle introduced by D'Alembert, and called by his name, but which is directly implied in Newton's Scholium on the third law of motion. This has been observed by many writers on Mechanics, but the connexion of Newton's Scholium with the modern theory of work and energy was first pointed out by Thomson and Tait; see their Treatise on Natural Philosophy, pp. 185–6.

70. The laws of Motion, like every law of nature, must ultimately depend for their establishment on their agreement with experiment and observation. Accounts of the different apparatus that have been devised for the purpose of verifying these laws will be found in the books especially devoted to the purpose, such as Ball's Experimental Mechanics. The most complete proof of the laws of motion, however, is derived from Physical Astronomy. The Lunar motions, for instance, have been calculated from equations depending solely on these laws; and the observed and calculated positions are found to agree with a precision that could only arise from the perfect accuracy of the principles from which they were deduced.

One of the simplest contrivances for illustrating the laws of motion, in the case of falling bodies, is that devised by Atwood, which we shall now proceed to consider.

71. **Atwood's Machine.**—In its simplest form this machine may be regarded as consisting of two masses connected by a string which passes over a small fixed pulley. We shall neglect the weight of the pulley, and also that of the string, as well as the friction at the axle of the pulley.



Suppose W and W' to represent the weights of the bodies, of which W is the greater.

Let T denote the tension of the string at any instant:

this tension, by the law of action and reaction, must act equally, and in opposite directions on the two bodies.

Accordingly, we may regard the body W as acted on by the pressure W downwards, and the tension T upwards; *i. e.* by the single force $W - T$ acting downwards—then, the corresponding acceleration f, from Art. 39, is given by the equation

$$f = \frac{W - T}{W} g.$$

Similarly, the upward acceleration of the other body is represented by $\frac{T - W'}{W'} g$.

Again, as the string is supposed *inextensible*, the velocities of the bodies at any instant are equal and opposite, and hence their accelerations also.

Accordingly we have

$$\frac{W - T}{W} = \frac{T - W'}{W'},$$

or

$$T = \frac{2WW'}{W + W'}. \tag{1}$$

This determines the tension of the string. Again, we have

$$W - T = W\frac{f}{g}, \quad T - W' = W'\frac{f}{g};$$

therefore

$$W - W' = (W + W')\frac{f}{g},$$

or

$$f = \frac{W - W'}{W + W'} g. \tag{2}$$

This determines the acceleration. By it the velocity and the space described in any time can be readily deduced.

The most important advantage of this apparatus is that, by taking bodies of nearly equal weights, we can make the acceleration $\frac{W - W'}{W + W'} g$ as small as we please.

A complete account of Atwood's apparatus is beyond the scope of this treatise. In a subsequent place we shall consider the modification required when allowance is made for the mass of the pulley.

EXAMPLES.

1. A mass of 488 grammes is fastened to one end of a chord which passes over a smooth pulley. What mass must be attached to the other end in order that the 488 grammes may rise through a height of 200 centimetres in 10 seconds, assuming $g = 980$ centimetres? *Ans.* 492 grammes.

2. Two weights of 14 and 18 ozs. are suspended by a fine thread which passes over a smooth pulley, if the system be free to move; find how far the heavier weight will descend in the first three seconds of its motion, and also the tension of the string. *Ans.* 18 feet, and $15\frac{3}{4}$ ozs.

72. Suppose that one of the bodies is placed on a *smooth* horizontal table, and that the string, by which the bodies are attached, passes over a smooth pulley placed at the edge of the table; then, denoting the tension of the string by T, we have, as before,

$$f = \frac{W - T}{W} g.$$

Again, since the motion of the body on the smooth table arises from the tension T, we have

$$f = \frac{T}{W'} g.$$

Eliminating T, we get

$$f = \frac{W}{W + W'} g. \qquad (3)$$

Again, equating the two values of f, we get

$$T = \frac{W W'}{W + W'}. \qquad (4)$$

It may be observed that the tension of the string in this case is half of that in Atwood's machine for the same masses.

73. **Masses on Two Smooth Inclined Planes.**—Suppose two bodies, of weights W and W', placed on two

planes, of inclinations i and i' to the horizon; and suppose the connecting string to lie in a vertical plane at right angles to the line of intersection of the two inclined planes, and to pass over a small pulley placed at the common summit of the planes: then, representing as before the tension of the string by T, since $W\sin i$ is the component of W acting parallel to the plane, we have

$$W\sin i - T = \frac{W}{g}f,$$

and $$T - W'\sin i' = \frac{W'}{g}f.$$

Hence $$f = \frac{W\sin i - W'\sin i'}{W + W'}g. \qquad (5)$$

Also $$T = \frac{WW'}{W+W'}(\sin i + \sin i'). \qquad (6)$$

It is evident that W or W' will descend according as $W\sin i$ or $W'\sin i'$ has the greater value.

The results of the two former Articles are particular cases of the preceding; and are, accordingly, cases of the formulæ (5) and (6). We shall next consider the preceding problems for rough planes.

74. **Motion on Uniformly Rough Planes.**—Suppose two bodies connected as in Art. 72, and let μ denote the coefficient of friction for the horizontal plane.

The friction acting against the motion of W' is represented by $\mu W'$; hence the pressure producing motion is $T - \mu W'$. We accordingly have the equation

$$T - \mu W' = \frac{W'}{g}f;$$

and also $$W - T = \frac{W}{g}f, \text{ as before.}$$

Hence we get

$$f = \frac{W - \mu W'}{W + W'} g, \tag{7}$$

and

$$T = \frac{WW'}{W + W'} (1 + \mu). \tag{8}$$

There can be no motion unless W is greater than $\mu W'$; as is also evident from elementary considerations.

Equation (7) may also be written in the form

$$\mu = \frac{W}{W'} - \frac{f}{g}\left(1 + \frac{W}{W'}\right); \tag{9}$$

from which μ can be determined when W and W' are known, f having been obtained by observation.

By this means the value of μ, the coefficient of dynamical friction, was obtained for several substances by Coulomb.

Again, let μ, μ' be the coefficients of dynamical friction for the inclined planes, in Art. 73.

Since the pressures on the planes are represented by $W \cos i$ and $W' \cos i'$, respectively, the corresponding frictions are $\mu W \cos i$ and $\mu' W' \cos i'$; consequently the total pressure acting on W, down the plane, is represented by

$$W(\sin i - \mu \cos i) - T;$$

and we get

$$W(\sin i - \mu \cos i) - T = \frac{W}{g} f.$$

And, similarly,

$$T - W'(\sin i' + \mu' \cos i') = \frac{W'}{g} f.$$

Hence, we have

$$f = \frac{W(\sin i - \mu \cos i) - W'(\sin i' + \mu' \cos i')}{W + W'} g, \tag{10}$$

and

$$T = \frac{WW'}{W + W'} \{\sin i + \sin i' + \mu' \cos i' - \mu \cos i\}. \tag{11}$$

EXAMPLES.

1. If the two equal masses in Atwood's machine be each 1 lb.; required the additional mass which, added to one of them, would generate a velocity of one foot in each mass at the end of the first second. *Ans.* $\frac{2}{g-1}$ lbs.

2. In the same case, find the tension of the string which connects the two masses. *Ans.* $\frac{g+1}{g}$ lbs.

3. Two smooth inclined planes are placed back to back: the inclination of one is 1 in 7, and of the other 1 in 10; a mass of 20 lbs. is placed on the first, and is connected by a string, with a mass of 30 lbs. placed on the second plane. Find the acceleration of the descent, and the tension of the string.

Ans. $f = \frac{g}{350}$, $T = 2\frac{32}{35}$ lbs.

4. A mass of 10 lbs., falling vertically, draws a mass of 15 lbs. up a smooth plane, of 30° inclination, by a string passing over a pulley at the top of the plane. Find the acceleration, the space fallen through in 10 seconds, and the tension of the string. *Ans.* $f = \frac{g}{10}$, $s = 5g$, $T = 9$ lbs.

5. A body, descending vertically, draws an equal body 25 feet in $2\frac{1}{2}$ seconds up a smooth plane, inclined 30° to the horizon, by means of a string passing over a pulley at the top of the plane. Determine the corresponding value of g.

Ans. 32.

6. Given the height, h, of a smooth inclined plane, find its length so that a given weight P, descending vertically, shall draw another given weight Q up the plane in the least possible time. *Ans.* $\frac{2Qh}{P}$.

7. A mass P, falling vertically, draws another, Q, by a string passing over a fixed pulley: if, at the end of t seconds, the connecting string be cut, find the height to which Q will ascend afterwards. *Ans.* $\left(\frac{P-Q}{P+Q}\right)^2 \frac{gt^2}{2}$.

8. A mass, hanging vertically, draws an equal mass along a rough horizontal plane. If at the end of one second the string be cut, find how far the mass will move along the plane before it is brought to rest by the friction.

Ans. $\frac{(1-\mu)^2 g}{8\mu}$.

9. In what time will a mass of 2 lbs., hanging vertically, draw a mass of 30 lbs. along a smooth horizontal table of 36 feet length? *Ans.* 6 seconds.

10. If the plane in the last example be rough, and the coefficient of friction be $\frac{1}{25}$, find the time occupied. *Ans.* $3\sqrt{10}$ seconds.

11. In the previous example, find at what instant during the motion the

string should be cut in order that the mass should just reach the edge of the table; and find the whole time of motion.

Ans. $12\sqrt{\frac{5}{13}}$ seconds, $\frac{3}{2}\sqrt{65}$ seconds ($q.p.$)

12. Two masses move on two smooth inclined planes, whose directions are at right angles to each other, and are connected by a string passing over the intersection of the planes. If the tension of the string be a maximum, find the inclination of either plane to the horizon. *Ans.* 45°.

13. In a single movable pulley, when there is equilibrium, the power and the weight hang by parallel strings. The weight being doubled, and the power halved, motion ensues. Prove that if the friction and inertia of the pulley be neglected, the tension of the string will be unaltered. (*Camb. Trip.*, 1874.)

14. In general, if P be the weight attached to the movable pulley, and Q that to the other end of the string, prove that the tension of the string during the motion is $\frac{3PQ}{P+4Q}$; and that the acceleration of the movable pulley is $\frac{2Q-P}{P+4Q}g$; the friction and inertia being neglected as before.

Let T denote the tension of the string, f the acceleration of P, and f' that of Q; and we have $2T - P = P\frac{f}{g}$, $Q - T = Q\frac{f'}{g}$; but $f' = 2f$; therefore, &c.

15. A train is travelling at a uniform rate on level rails. W is the weight of the fore portion of the train, and W' that of the brake-van at the end of the train. If the brakes be applied to the brake-van, find the stress produced on the couplings between it and the next carriage, assuming μ to represent the coefficient of friction.

Ans. $\mu\frac{WW'}{W+W'}$.

16. In Atwood's machine, if the descending weight be a rigid homogeneous vertical rod AB, prove that the longitudinal stress at any point P of the rod, during the motion, is represented by $\frac{BP}{AB}T$, where T is the tension of the string.

17. In Atwood's machine if the pulley be rough, and if the effect of friction be to prevent motion until the tension of the string at one end be greater than that at the other by $\frac{1}{n}$th of the latter tension, prove that the effect on the acceleration will be the same as if the pulley remained smooth and the smaller weight were increased by $\frac{1}{n}$th.

18. In Atwood's machine, a mass P is attached to one end of the string, and two masses, Q and R, to the other end, where $Q + R > P$, and $P > Q$. After the united masses Q and R have descended s feet from rest, R is detached: find how much further Q will move before being brought to rest.

Let f be the acceleration in the first stage of the motion, f' that in the second, v the velocity at the instant R is detached, x the required distance; then

$$2fs = v^2 = 2f'x;$$

therefore
$$x = \frac{f}{f'}s = \frac{\overset{R}{\cancel{R}} + Q - \overset{P}{\cancel{R}}}{P + Q + R} \cdot \frac{P+Q}{P-Q}s.$$

CHAPTER IV.

IMPACT AND COLLISION.

75. **Collision of Homogeneous Spheres.**—In this chapter it is proposed to consider some elementary cases of impact of solids, but principally the collision of homogeneous spherical bodies, moving without rotation, whose centres, at the instant of collision, move in right lines lying in the same plane (all friction being neglected).

There are two cases to be considered, according as the centres of the spheres move in the same or in different right lines. The former is called direct, the latter oblique collision.

We commence with the former case, and at first suppose the centres to move in the same direction along the line.

76. **Direct Collision.**—Let M and M' represent the masses of the bodies, V and V' their velocities before, v and v' those after, collision. We also suppose M to impinge on M'.

The whole impact may be divided into two stages. During the first, the bodies compress each other, and the impinging body M, moving with a greater velocity than the other, accelerates its motion, until the exact instant at which their mutual compression is the greatest, and when they are moving with a common velocity. During the second stage, the bodies tend to revert to their original shape, and the forces thus brought into play, called the *forces of restitution*, tend to cause the bodies to separate by still further diminishing the velocity of the impinging body, and increasing that of the other.

Suppose u represents the common velocity at the instant of greatest compression; then the quantity of motion lost by M during the first stage of the shock is $M(V-u)$, and that gained by M' is $M'(u-V')$.

These are the measures, by Art. 66, of the entire mutual forces which act during this stage of the collision; and, by the Third Law of Motion, they must be equal and opposite.

Hence, we have

$$M(V-u) = M'(u-V'),$$

or

$$u = \frac{MV + M'V'}{M+M'}. \tag{1}$$

In the case of perfectly *non-elastic* bodies, in which no forces of restitution are brought into play, the bodies would proceed after collision to move with this common velocity.

There is probably no case in nature of a perfectly non-elastic solid. All solid bodies with which we are acquainted have a tendency to recover, in different degrees, their original forms after being compressed. This tendency arises from their *elasticity*.

Bodies are said to be *perfectly elastic* when the forces of restitution, brought into play during the second stage, are exactly equal to the forces of compression, which act during the first. In this case the impinging body M would lose a further quantity of motion, $M(V-u)$, equal to that which it lost in the first stage. Therefore its velocity v, after the shock, will be equal to $2u - V$.

In like manner we have $v' = 2u - V'$. Thus we are enabled to determine the velocities, v, v', after direct collision, in the case of perfectly elastic bodies.

Bodies are in general, however, imperfectly elastic; that is, the whole force of restitution is less than that of compression. The Law of restitution, as derived from experiment, is usually stated thus:—*The ratio which the whole force of restitution bears to the force of compression is constant while the impinging substances remain the same.* This ratio is usually represented by the letter e, having been by many writers called the modulus or *coefficient of elasticity;* but as this title is now employed in a different sense, we shall follow the current usage in adopting the name, *coefficient of restitution.*

From this law it follows that the quantity of motion lost by M during the second stage of the impact bears a constant ratio to that lost during the first; and similarly for the quantity of motion gained by M'.

Accordingly

$$M(u-v) = eM(V-u), \quad M'(v'-u) = eM'(u-V').$$

Hence we get

$$v' - v = e(V - V'), \tag{2}$$

and

$$MV + M'V' = Mv + M'v'. \tag{3}$$

These equations enable us to determine the velocities, v, v', after impact, when those before impact are given, as also the masses M, M', and the coefficient of restitution.

It should be observed that equation (3) expresses that the total quantity of motion of the two bodies is the same after impact as before. This result is a particular case of a general principle which shall be subsequently considered (*see* Art. 81).

The result contained in (2) may be stated thus—*In direct collision** *of two spheres, the relative velocity after collision bears a constant ratio to the relative velocity before collision.*

This law was established by Newton, as the result of experiment (see his *Leges Motus*, scholium), and the coefficients of restitution for several substances, such as glass, ivory, steel, &c., were determined by him.

In more recent times a number of careful experiments were undertaken by Mr. Hodgkinson on the laws of restitution. The results are to be found in the Report of the British Association, 1834, and also in the Transactions of the Royal Society. His conclusions agree in the main with the law laid down by Newton, given above.

Some of the more important results of Mr. Hodgkinson's experiments may be briefly stated as follows:—

The coefficient of restitution diminishes slowly as the velocity of impact increases; it is independent of the relative magnitude of the masses. In impact between bodies differing much in *hardness*, the coefficient of restitution is nearly equal to that between two specimens of the softer body.

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* This result is by some writers taken as the basis of the rational theory of collision. However, the method here given is that more usually adopted; it has the great advantage of connecting the problem directly with the consideration of force, and of illustrating the principle (Art. 66) that impulsive forces must be regarded as forces of great intensity whose time of action is very short.

No perfectly elastic body exists in nature: glass, however, may be regarded as nearly so, its coefficient being $\frac{15}{16}$, approximately, as determined by Newton.

If the mass M' be supposed at rest, and very great in comparison with M, v' becomes very small, and we have approximately $v = -eV$.

Hence, if a body impinge perpendicularly, with a velocity V, upon a fixed plane, it will return back in its former direction of motion with a velocity represented by eV, where e is the coefficient of restitution.

77. **Height of Rebound.**—If a body fall from a height h on a fixed horizontal plane, then V, the velocity with which it strikes the plane, is equal to $\sqrt{2gh}$. The velocity of rebound is equal to eV, or $e\sqrt{2gh}$: hence, if h' be the height to which it rebounds, we have

$$\sqrt{2gh'} = e\sqrt{2gh},$$

or

$$h' = e^2h. \qquad (4)$$

In all cases of collision the student should be careful to give the proper algebraic signs to the velocities. The velocity V of the mass M is usually taken as positive; and hence the other velocities will have positive or negative signs, according as they take place in the same or the opposite direction to that of V.

Examples.

1. If a mass M impinge directly on another mass M', at rest; find the relation between them when the impinging mass is reduced to rest by the collision.

Ans. $M = eM'$.

2. A ball of 6 lbs. mass, moving at the rate of 10 miles an hour, overtakes another of 4 lbs. mass, moving at 5 miles an hour: determine their velocities after collision, assuming $e = \frac{1}{2}$—the impact being supposed direct.

Ans. $v = 7$; $v' = 9\frac{1}{2}$.

3. Find the corresponding velocities in the case when the balls are moving in opposite directions.

Ans. $v = 1$, $v' = 8\frac{1}{2}$.

4. A mass of 50 lbs., moving at the rate of 10 feet per second, overtakes another mass of 25 lbs., moving at the rate of 6 feet per second; if both masses be perfectly elastic, find their velocities after the shock.

Ans. $v = 7\frac{1}{3}$; $v' = 11\frac{1}{3}$.

5. A sphere impinges directly on a sphere of the same mass. If they be both perfectly elastic, prove that they interchange velocities, after collision.

6. A mass drops from a height of 25 feet above a fixed horizontal plane, and rebounds to a height of 9 feet; find the coefficient of restitution. $e = \frac{3}{5}$.

7. A mass of 10 lbs., moving with a velocity of 10 feet per second, impinges directly on another of 5 lbs., supposed at rest. If the coefficient of restitution be $\frac{4}{5}$, and the duration of collision be $\frac{1}{100}$th part of a second, determine the *mean value* of the mutual pressure between the balls during the collision.

Here, we easily find $v' = 12$. Hence, by Art. 39, we find that the pressure which, acting for $\frac{1}{100}$th of a second, would generate a velocity of 12 feet per second, in a body of 5 lbs. mass, is $187\frac{1}{2}$ lbs.

8. An imperfectly elastic sphere falls from a given altitude above a horizontal plane, and rebounds continually: find the whole space described; and also the whole time before it is brought to rest; neglecting the time occupied by the series of impacts, and also the resistance of the air.

Let a denote the given altitude, and s the whole space described.

Then the height of the first rebound is ae^2; that of the second ae^4, &c.

Therefore $$s = a + 2ae^2 + 2ae^4 + \&c. = a\,\frac{1+e^2}{1-e^2}.$$

Again, if V, V_1, V_2, &c., V_n, be the velocities with which the sphere strikes the plane, at the different impacts, we have

$$V_1 = eV,\quad V_2 = eV_1 = e^2V,\quad \&c.,\quad V_n = e^nV.$$

Also let t, t_1, t_2, &c., t_n, be the corresponding intervals of time; then

$$t = \frac{V}{g},\quad t_1 = \frac{2V_1}{g} = \frac{2V}{g}e,\quad t_2 = \frac{2V}{g}e^2,\ \&c.$$

Hence, the entire time T is given by the equation

$$T = \frac{V}{g}(1 + 2e + 2e^2 + \&c.) = \frac{V}{g}\,\frac{1+e}{1-e}.$$

78. **Oblique Collision.**—We now proceed to the case of oblique collision; *i. e.* where the centres of the spheres are not moving in the same right line.

We shall suppose the spheres to be homogeneous, and perfectly smooth; so that their entire mutual action and reaction has place along the common normal at their point of contact, which is the line connecting the centres of the spheres; and that the lines along which the centres of the spheres are moving before collision lie in the same plane.

Let V, V' be the velocities at the instant of impact; and α, α' the angles which the line joining the centres, at the commencement of the collision, makes with the respective directions of motion.

Let v, v'; β, β' be the corresponding velocities and angles at the end of the collision.

Resolve V into its components, $V\cos\alpha$ and $V\sin\alpha$, respectively along and perpendicular to the right line joining the centres. Make a similar resolution of the velocities after collision. Then, since the forces brought into play during the collision act along the line joining the centres, the velocities perpendicular to that line are unaffected by the collision.

Hence we have

$$V\sin\alpha = v\sin\beta, \quad V'\sin\alpha' = v'\sin\beta'. \tag{5}$$

Again, the component velocities $V\cos\alpha$, &c., along the line joining the centres of the spheres, will, by the Laws of Motion, be subject to the same relations (2), (3), as those already established for direct collision—hence, we obtain the two additional equations

$$v'\cos\beta' - v\cos\beta = e(V\cos\alpha - V'\cos\alpha'). \tag{6}$$

$$MV\cos\alpha + M'V'\cos\alpha' = Mv\cos\beta + M'v'\cos\beta'. \tag{7}$$

These, along with the two preceding equations (5), are sufficient for the determination of the velocities and the directions of motion after impact, when the corresponding velocities and directions before impact are known; as also the masses and the coefficient of restitution.

In the case of oblique collision of a sphere against a smooth fixed plane, let V be the velocity of the sphere before collision, and α the angle its direction of motion makes with the perpendicular to the smooth plane; then $V\cos\alpha$ represents the velocity perpendicular, and $V\sin\alpha$ that parallel, to the fixed plane.

If v and β represent the corresponding velocity and angle after collision, since the velocity parallel to the smooth ~~place~~ plane is unaltered by collision, we have

$$v\sin\beta = V\sin\alpha. \tag{8}$$

Again, the velocity perpendicular to the plane will be affected in the same manner as in direct collision, and we accordingly have

$$v \cos\beta = eV\cos\alpha. \tag{9}$$

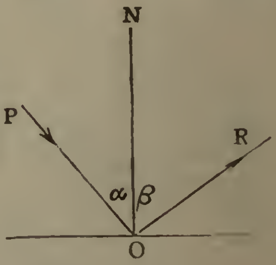
Hence, by division, we get

$$\tan\alpha = e\tan\beta, \tag{10}$$

which gives the direction of motion after impact.

The angles α, β are sometimes called the angles of incidence and reflexion; and the preceding result shows that the tangents of these angles are to each other in the constant ratio of the coefficient of restitution to unity. These angles are equal in the case of perfectly elastic bodies.

The subsequent motion of the body depends on the continuous forces which act on it. Under gravity only the path is a parabola, as in Art. 46; and the parabolic path is determined from the initial velocity v, and the initial direction of motion, β.



EXAMPLES.

1. If the mass M' be at rest before collision, find the directions of motion after collision.

Ans. $\beta' = 0$; $\tan\beta = \dfrac{M + M'}{M - eM'}\tan\alpha$.

2. A perfectly elastic ball impinges obliquely on another of equal mass at rest; prove that the directions of their motions after impact are at right angles to one another.

3. A ball impinges on another at rest; prove that if the coefficient of restitution be equal to the ratio of their masses the balls will move in directions at right angles to each other, whatever be the direction of the impact.

4. How is this statement to be modified in the case of direct collision?
The impinging ball is brought to rest by the collision.

5. A ball is reflected in succession by two fixed smooth planes of the same substance, which are at right angles to one another; the ball moves in a plane at right angles to the intersection of the fixed planes. Prove that the direction of motion before the first and after the second reflection are parallel.

6. A projectile strikes a perfectly elastic wall, which is perpendicular both to the horizon and to the plane of the projectile's flight: find the horizontal range of the reflected projectile.

Since the angles of incidence and reflection are equal in this case, as also the velocities before and after impact, it is evident that the parabolic path of the projectile after striking the wall is equal in every respect to that which it would have continued to describe if there had been no wall interposed. Accordingly the problem is solved by aid of Art. 48.

7. An imperfectly elastic particle is projected from a point in a smooth horizontal plane, with a given velocity V, and in a given direction α, and proceeds to describe a series of parabolic paths by a number of rebounds from the plane: find the whole time elapsed before it ceases to rebound; and also its subsequent motion.

Resolve the velocity of projection into vertical and horizontal components, $V\sin\alpha$ and $V\cos\alpha$.

The horizontal component $V\cos\alpha$ will be unaltered by the successive impacts, and accordingly remains constant throughout the motion: the vertical component $V\sin\alpha$ will be altered at each impact in the same manner as in direct collision; accordingly it may be treated as in Ex. 8, p. 68.

Hence if T be the entire time before the vertical velocity $V\sin\alpha$ is destroyed, we easily get, as in the example referred to,

$$T = \frac{2V\sin\alpha}{g}\,\frac{1}{1-e}$$

Again, since the horizontal velocity is constant, and equal to $V\cos\alpha$, the whole range before the vertical velocity ceases is

$$\frac{V^2\sin 2\alpha}{g(1-e)}.$$

The body would subsequently move along the plane with the constant velocity $V\cos\alpha$.

It should be observed that the particle, in this problem, describes a series of parabolic curves, one for each rebound.

79. **Vis Viva of a System.**—If any mass m be moving without rotation, and if v denote, at any instant, the velocity common to all its points, then the quantity represented by mv^2 is called the *vis viva* of the mass at the instant. In general, in the motion of any system of masses, if each element of mass be multiplied by the square of its velocity at any instant, and the sum of these products taken for the entire system, this sum is called the *vis viva of the system* at that instant. It is represented by the expression

$$\Sigma(mv^2)$$

In perfectly elastic Spheres the Vis Viva is unaltered by Collision.

Since $e=1$ in this case, equation (2) becomes $V+v=V'+v'$;

also, we have $\quad M(V-v) = M'(v'-V')$.

Multiplying we get

$$M(V^2 - v^2) = M'(v'^2 - V'^2),$$

or

$$MV^2 + M'V'^2 = Mv^2 + M'v'^2. \qquad (11)$$

Hence, in direct collision, the total *vis viva* is the same after collision as before.

It can be easily seen that the same principle holds in the oblique collision of perfectly elastic spheres. For the preceding demonstration holds for the components of velocity estimated along the line joining the centres of the spheres, at the instant of collision: moreover, the tangential components of velocity are unaffected by collision; consequently (since $V^2 = V^2\sin^2 a + V^2\cos^2 a$, &c.) it follows that the total *vis viva* is unaltered by collision, in the case of perfect elasticity.

80. **Momentum of any System in Motion.**—Let (x, y, z), (x', y', z'), (x'', y'', z''), &c., at any instant, denote the coordinates of a number of moving particles, m, m', m'' &c., referred to a fixed system of rectangular axes; then, by Art. 12, the component velocities of m, at the instant, parallel to the axes of x, y, z, are $\dot{x}$, $\dot{y}$, $\dot{z}$, respectively.

Again, resolving the quantity of motion of m, in the same directions, we get for its components, the expressions

$$m\dot{x},\ m\dot{y},\ m\dot{z}, \text{ or } m\frac{dx}{dt},\ m\frac{dy}{dt},\ m\frac{dz}{dt}.$$

If this be done for the other masses m', m'', &c., the whole quantity of motion or *momentum of the system, at the instant, estimated parallel to the axis of* x is represented by $\Sigma m\dot{x}$.

In like manner, the whole momentum parallel to the axes of y and z are $\Sigma m\dot{y}$ and $\Sigma m\dot{z}$, respectively.

Again, let $\bar{x}$, $\bar{y}$, $\bar{z}$ represent the coordinates of the common centre of gravity of the system m, m', &c., at the same instant, we have, denoting the sum of the masses by M,

$$M\bar{x} = \Sigma mx,\quad M\bar{y} = \Sigma my,\quad M\bar{z} = \Sigma mz.$$

Moreover, since these equations hold throughout the motion, we may differentiate them, with respect to the time t, and thus we obtain the equations

$$\left.\begin{aligned} M\frac{d\bar{x}}{dt} &= \Sigma m\frac{dx}{dt} \\ M\frac{d\bar{y}}{dt} &= \Sigma m\frac{dy}{dt} \\ M\frac{d\bar{z}}{dt} &= \Sigma m\frac{dz}{dt} \end{aligned}\right\}. \qquad (12)$$

Hence the resolved part of the momentum of a system, in any direction, is equal to the whole mass of the system multiplied into the component of the velocity of the centre of gravity, in the same direction.

8. **Conservation of Momentum.**—It is easily seen that the Momentum in any direction of any system of bodies is unaltered by their mutual collision. For, under all circumstances of collision, the actions and reactions are equal and opposite; and as these forces are measured by the quantities of motion which they generate or destroy, it follows, if two bodies of the system come into collision, that whatever momentum, in any direction, is generated by the impact in one of the impinging bodies, an equal momentum in the same direction must be destroyed in the other. So that the entire momentum, in that direction, is unaltered by the collision. The same holds whatever number of collisions be supposed to take place between the members of the system.

Hence, we infer that the entire momentum of the system, resolved in any direction, is unaffected by any impacts among the parts of the system.

It can be seen, without difficulty, that the same mode of reasoning applies to any case of internal mutual action between the several parts of the system, whether arising from attractions, molecular forces, or otherwise: since, in all cases, to every action corresponds an equal and opposite reaction.

We accordingly infer that, if a *system be only subjected to the internal mutual forces between the bodies which constitute it,* *the total resolved momentum in any direction is constant;* i.e. $\Sigma m\dot{x}$, $\Sigma m\dot{y}$, $\Sigma m\dot{z}$, have constant values during the motion.

82. **Conservation* of Motion of Centre of Gravity.**—It follows from (12) that $\frac{d\bar{x}}{dt}$, $\frac{d\bar{y}}{dt}$, $\frac{d\bar{z}}{dt}$, i.e. the component velocities of the centre of gravity of a system, will be constant throughout the motion whenever the quantities of motion $\Sigma m\dot{x}$, $\Sigma m\dot{y}$, $\Sigma m\dot{z}$, are constant.

Hence, from the preceding Article, it follows that the velocity, and also the direction of motion of the centre of gravity of any system, are constant, whenever the system is only subject to the mutual actions and reactions of the bodies which constitute it.

This is a generalization of the principle of inertia contained in the First Law of Motion, and may be otherwise stated thus:—*A system of bodies cannot by their mutual actions and reactions alter the motion of their common centre of gravity.*

Hence, in such a system, when not acted on by any external forces, the common centre of gravity must either remain at rest or move uniformly in a right line.

83. The general principle, that the entire quantity of motion of two or more bodies is unaltered by their mutual actions and reactions, furnishes us with a ready method of solving some elementary problems.

For example, suppose two masses, m and m', to be connected by a string, and laid on a perfectly smooth horizontal table, at a distance from each other less than the length of the string. Now, let a given impulse be applied to m along the line which joins it to m', the motion which ensues after the string becomes stretched, can be easily found as follows:

Let mV be the quantity of motion communicated to m by the impulse, then, after the string becomes tight, the bodies must move with a common velocity. Let v_1 denote this common velocity; then, since the whole quantity of motion of the two bodies remains the same, we have

$$(m + m')v_1 = mV.$$

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* This proof corresponds in the main with that given by Newton. See Leges Motus, Cor. iv.

Consequently they move along the line with a common velocity

$$\frac{mV}{m+m'}.$$

In this problem we have supposed the motion one of pure sliding; and we neglect the weight of the string in it as also in the next problem.

84. *A mass M, after falling through a height h, from the edge of a smooth table, commences to draw by an inextensible string another mass M', which rests on the table; to find the velocity communicated to M' at the instant that the string becomes tightened, and also the impulsive tension of the string.*

The velocity acquired by M, in consequence of its fall, is represented by $\sqrt{2gh}$; and since at the end of the impulsive strain the bodies are moving with equal velocities, and also the quantity of motion is unaltered by the impulsive action, we must have

$$(M+M')v_1 = MV = M\sqrt{2gh},$$

or

$$v_1 = \frac{M}{M+M'}\sqrt{2gh}, \qquad (13)$$

where v_1 denotes the common velocity at the instant in question.

Again, the impulsive tension of the string is measured by the quantity of motion communicated to M'; and accordingly is represented by

$$\frac{MM'}{M+M'}\sqrt{2gh}.$$

If the table be rough, since the friction of the table is proportional to the weight of M', it may be neglected in comparison with the impulsive force, and we obtain the same value for v_1, as in the case of a smooth table.

Again, if the string be extensible, equation (2) gives the velocity of M' at the instant that M and M' are moving with the same velocity. The subsequent motion is determined, as in Art. 72, for a smooth, and Art. 74 for a rough, table.

EXAMPLES.

1. A sphere, of 30 lbs., moving with a velocity of 45 feet per second, overtakes another, of 27 lbs., moving 32 feet per second; if the relative coefficient of elasticity be $\frac{2}{3}$, find their velocities after collision. *Ans.* $34\frac{14}{19}$, $43\frac{23}{57}$.

2. Two spheres meet directly with equal velocities; find the ratio of their masses that one of them, M, should be reduced to rest by the collision—(1) when perfectly elastic; (2) for elasticity e. *Ans.* (1) $M = 3M'$, (2) $M = M'(1 + 2e)$.

3. If two equal and perfectly elastic spheres be dropped at the same instant from different heights, h and h', above a horizontal plane; determine whether their common centre of gravity will ever rise to its original height.

Ans. No, unless $\sqrt{\frac{h}{h'}}$ is a *rational* quality.

4. A 10 lb. shot is fired from a gun of 12 cwt., that is quite free to move. The velocity with which the shot leaves the mouth of the gun is 1600 feet per second; find the velocity of the gun's recoil. *Ans.* 11·9 feet per second.

5. Three homogeneous spherical bodies, m, m', m'', are placed with their centres in a row. If m be projected with a given velocity V towards m', to find the magnitude of m' in order that the velocity communicated to m'' by its intervention shall be the greatest possible.

Let e, e' denote the relative coefficients of restitution between m, m', and between m', m'', respectively. Then, if v' be the velocity of m' after the first collision, we get from (2), (3) Art. 76,

$$v' = \frac{m(1+e)V}{m+m'}.$$

In like manner, if v'' be the velocity of m'' after the second collision, we have

$$v'' = \frac{m'(1+e')v'}{m'+m''} = \frac{mm'(1+e)(1+e')V}{(m+m')(m'+m'')}.$$

Accordingly, $\frac{m'}{(m+m')(m'+m'')}$ must be a maximum; or $\frac{(m+m')(m'+m'')}{m}$ is a minimum: i. e. $m + m' + m'' + \frac{mm''}{m'}$ is a minimum, or $m' + \frac{mm''}{m'}$ is a minimum: hence, $m' = \sqrt{mm''}$, by elementary algebra, consequently the masses must be in geometrical progression.

This reasoning is readily extended to the case of any number of spheres placed in a row; and, when the first and last are given, the masses must be in geometrical progression in order that the velocity communicated to the last should be the greatest possible.

6. Two particles are connected by a string, and laid on a uniformly rough horizontal table, at a distance from each other less than the length of the string. One of the particles receives a given impulse along the line joining them: determine the motion which ensues after the tightening of the string.

7. Find an expression for the *vis viva* lost in the direct collision of two imperfectly elastic spheres.

From equations (2), (3), Art. 76, we have

$$(mV + m'V')^2 = (mv + m'v')^2,$$

and

$$mm'e^2(V - V')^2 = mm'(v - v')^2.$$

This latter may be written

$$mm'(V - V')^2 = mm'(v - v')^2 + (1 - e^2)mm'(V - V')^2.$$

Hence, by addition,

$$(m + m')(mV^2 + m'V'^2) = (m + m')(mv^2 + m'v'^2) + (1 - e^2)mm'(V - V')^2.$$

Therefore $$mV^2 + m'V'^2 = mv^2 + m'v'^2 + (1 - e^2)\frac{mm'}{m + m'}(V - V')^2.$$

Accordingly, the *vis viva* lost by the collision is represented by

$$(1 - e^2)\frac{mm'}{m + m'}(V - V')^2.$$

8. Find the loss of *vis viva* caused by the direct impact of two balls, one weighing 10 lbs. and falling from a height of 20 feet, the other at rest and weighing 30 lb.; assuming the coefficient of restitution $= \frac{1}{2}$.

Ans. $\frac{9}{16}$th of the original *vis viva.*

9. A body, after sliding down a smooth inclined plane of given height, rebounds from a hard horizontal plane; find the range on the latter plane.

10. A mass M, after falling freely through h feet, begins to pull up a heavier mass M_1 by means of a string passing over a pulley, as in Atwood's machine; to find the height through which it will lift it.

Let v_1 be the velocity communicated to M_1 by the impulsive action; then by Art. 84, we have $v_1 = \frac{M}{M + M_1}\sqrt{2gh}$.

During the subsequent motion, as in Art. 71, M_1 is subject to a uniform retardation $\frac{M_1 - M}{M_1 + M}g$; accordingly, if H denote the height through which M_1 ascends before it is brought to rest, we have

$$H = \frac{v_1^2}{2f} = \frac{M^2}{M_1^2 - M^2}h.$$

11. An inelastic particle falls from rest to a fixed inclined plane, and slides down the plane to a fixed point in it; show that the locus of the starting point is a straight line when the time to the fixed point is constant. (*Camb. Trip.*, 1871.)

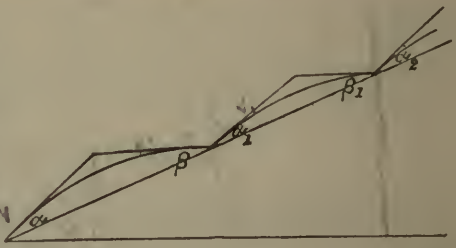
12. Two equal balls of radius a are in contact and are struck simultaneously by a ball of radius c moving in the direction of their common tangent; if all the balls be of the same material, the coefficient of elasticity being e, find the velocities of the balls after impact, and prove that the impinging ball will be reduced to rest if

$$2e = \frac{c^2(a+c)^2}{a^3(2a+c)}. \qquad \text{(Camb. Trip., 1871.)}$$

13. Show how to determine the motion of two elastic spheres after direct impact, and prove that the relative velocity of each of them with regard to the centre of mass of the two is, after the impact, reversed in direction and reduced in the ratio $e : 1$, e being the coefficient of restitution.

A series of n elastic spheres whose masses are 1, e, e^2, &c., are at rest, separated by intervals, with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Prove that the final *vis viva* of the system is $(1 - e + e^n)u^2$. (*Ibid.*, 1875.)

14. An elastic ball makes a series of rebounds from a perfectly smooth inclined plane: to investigate its motion.



Let i be the inclination of the plane to the horizon, and suppose the ball projected from the point O in the plane, in a direction which makes the angle α with the plane. Let β, β_1, &c., β_n be the angles at which the ball strikes the plane at the first, second, ... n^{th} impacts; and α_1, α_2, . . . α_n, the angles it makes after rebounding.

Then by equation 9, s. 54, we have

$$\cot\beta = \cot\alpha - 2\tan i;$$

but by (10) Art. 78,

$$\cot\beta = e\cot\alpha_1,$$

$$\therefore \quad e\cot\alpha_1 = \cot\alpha - 2\tan i.$$

Similarly

$$e\cot\alpha_2 = \cot\alpha_1 - 2\tan i;$$

$$\therefore \quad e^2\cot\alpha_2 = \cot\alpha - 2(1+e)\tan i,$$

and it is easily seen that we have, in general,

$$e^n\cot\alpha_n = \cot\alpha - 2(1 + e + \ldots + e^{n-1})\tan i$$

$$= \cot\alpha - \frac{2(1-e^n)}{1-e}\tan i,$$

from which the angle after the n^{th} rebound can be found.

Again, the ball will proceed to bound up the plane so long as the angles α_1, α_2, . . . are each less than $90° - i$.

If α_n be the first of the series of angles which exceeds $90° - i$, we will have $\cot \alpha_n < \tan i$.

If $\cot \alpha$ is greater than $\frac{2 \tan i}{1 - e}$ it can be readily shown that for all values of n, α_n is less than $90° - i$; and in this case accordingly the ball would proceed to ascend the plane by an indefinite series of parabolic paths.

But if $\cot \alpha$ be less than $\frac{2 \tan i}{1 - e}$, after a certain number of impacts, the body would proceed to rebound *down* the inclined plane.

In the particular case where $\cot \alpha = \frac{2 \tan i}{1 - e}$, or $2 \tan i = \cot \alpha (1 - e)$, we have

$$e \cot \alpha_1 = e \cot \alpha; \quad \therefore \ \alpha_1 = \alpha;$$

hence

$$\alpha = \alpha_1 = \alpha_2 = \ldots = \alpha_n;$$

or, all the angles of rebound are equal to one another; consequently the series of parabolic paths in this case are similar, and the particle would proceed up the plane with an indefinite number of rebounds.

In general, let $t_1, t_2, \ldots t_n$ be the times of flight for the series of parabolic paths, and $v_1, v_2, \ldots v_n$, the velocities of the successive rebounds, then by equation (5), Art. 49, we have

$$t_1 = \frac{2v \sin \alpha}{g \cos i}, \quad t_2 = \frac{2v_1 \sin \alpha_1}{g \cos i}, \ \&c.$$

But if v' be the velocity with which the ball strikes the plane at the first impact, we have

$$v' \sin \beta = v \sin \alpha;$$

but by Art. 78,

$$v_1 \sin \alpha_1 = ev' \sin \beta = ev \sin \alpha;$$

consequently

$$t_2 = et_1; \text{ also } t_3 = et_2 = e^2 t_1, \ \&c.$$

Hence the times of flight are in geometrical progression, having e for their common ratio.

If the intervals of time occupied by the successsive impacts be neglected, we get for the time T, of describing the first n parabolas,

$$T = \frac{2v \sin \alpha}{g \cos i} \frac{1 - e^n}{1 - e}.$$

Again, let $R_1, R_2, \ldots R_n$ denote the consecutive ranges on the inclined plane; then, by Art. 49, we have

$$R_1 = \tfrac{1}{2} g t_1^2 \frac{\cos(\alpha + i)}{\sin \alpha} = \tfrac{1}{2} g t_1^2 \cos i (\cot \alpha - \tan i).$$

Similarly,

$$R_2 = \tfrac{1}{2} g t_2^2 \cos i (\cot \alpha_1 - \tan i) = \tfrac{1}{2} g t_1^2 \cos i (e^2 \cot \alpha_1 - e^2 \tan i)$$

$$= \tfrac{1}{2} g e t_1^2 \cos i \{\cot \alpha - (2 + e) \tan i\}.$$

And, in general,

$$R_n = \tfrac{1}{2} g t_n^2 \cos i (\cot \alpha_{n-1} - \tan i)$$

$$= \tfrac{1}{2} g e^{n-1} t_1^2 \cos i \{\cot \alpha - (2 + 2e + \ldots + 2e^{n-2} + e^{n-1}) \tan i\}$$

$$= \tfrac{1}{2} g e^{n-1} t_1^2 \cos i \cot \alpha - \tfrac{1}{2} g e^{n-1} t_1^2 \sin i \left(\frac{2 - e^{n-1} - e^n}{1 - e}\right)$$

$$= \tfrac{1}{2} g t_1^2 e^{n-1} \cos i \cot \alpha - g \frac{e}{1 - e} t_1^2 \sin i + \tfrac{1}{2} g \frac{1 + e}{1 - e} e^{2n-2} t_1^2 \sin i.$$

Hence the sum of n ranges is found to be

$$= 2 \frac{v^2}{g} \frac{\sin^2 \alpha}{\cos i} \frac{1 - e^n}{1 - e} \left\{\cot \alpha - \frac{1 - e^n}{1 - e} \tan i\right\}$$

$$= vT \sin \alpha \left(\cot \alpha - \frac{1 - e^n}{1 - e} \tan i\right).$$

If $\cot \alpha$ be greater than $\dfrac{2 \tan i}{1 - e}$ we get the entire range on the inclined plane by making $n = \infty$; hence the entire range is, in this case,

$$vT \sin \alpha \left\{\cot \alpha - \frac{\tan i}{1 - e}\right\}.$$

The preceding question was discussed at great length by Bordoni, *Mem. della Societa Ital.*, 1816. See also Walton's *Problems*, pp. 262, 263, 3rd edition.

15. In the preceding example show that the greatest distances of the body from the inclined plane in the successive parabolic paths are in geometrical progression, having e^2 as their common ratio.

16. If two bodies, of the same elasticity, be projected with the same velocity from a point on an inclined plane, and if the directions of projection make equal angles at opposite sides of the perpendicular to the plane, prove that the series of parabolic paths described, one up, the other down the plane, will be described in times which are respectively equal in pairs.

☞ 17. An imperfectly elastic ball falls from a height, h, upon an inclined plane; find the range between the first and second rebounds. *Ans.* $4eh \sin i\ (1 + e)$.

18. Prove that, in order to produce the greatest deviation in the direction of a smooth billiard ball of diameter a by impact on another equal ball at rest, the former must be projected in a direction making an angle $\sin^{-1} \dfrac{a}{c} \sqrt{\dfrac{1 - e}{3 - e}}$ with the line (of length c) joining the two centres; e being the coefficient of elasticity.
Camb. Trip., 1873.

19. A bucket and a counterpoise, connected by a string passing over a pulley, just balance one another, and an elastic ball is dropped into the centre of the bucket from a distance h above it; find the time that elapses before the ball

ceases to rebound, and prove that the whole descent of the bucket during this interval is $\frac{4mh}{2M+m}\,\frac{e}{(1-e)^2}$, where m, M are the masses of the ball and the bucket, and e is the coefficient of restitution. *Camb. Trip.*, 1875.

Let v be the velocity of the ball just before the first impact. The relative velocity after the first impact is ev, and the relative acceleration is g, since the acceleration of the bucket is zero.

Therefore the time during which the ball rebounds is

$$\frac{2v}{g}(e+e^2+e^3+\ldots)=\frac{2v}{g}\,\frac{e}{1-e}=2\,\frac{e}{1-e}\sqrt{\frac{2h}{g}}.$$

Let V_1, V_2, V_3, . . . be the velocities of the bucket between the first, second, third, . . . impacts.

Then $$V_1=\frac{m(1+e)}{2M+m}\,v,\quad V_2=V_1+\frac{m(1+e)}{2M+m}\,ev,\ \text{\&c.},$$

and the space described by the bucket is

$$\frac{2v}{g}(eV_1+e^2V_2+e^3V_3+\ldots)=\frac{2me\,v^2}{g\,(2M+m)\,(1-e)^2}=\frac{4mh}{2M+m}\,\frac{e}{(1-e)^2}.$$

(This proof is taken from Greenhill's solutions of Cambridge Problems and Riders for 1875.)

20. A particle is projected with a velocity V, in a direction making an angle α with the horizon, and strikes a vertical wall, at a distance a from the point of starting. Find when and where it will strike the horizontal plane drawn through its initial position.

Ans. $T=\frac{2V\sin\alpha}{g}$. The distance from the wall at which it will strike the ground $=e\left(\frac{V^2\sin 2\alpha}{g}-a\right)$, where e is the coefficient of restitution for the particle and the wall.

21. A large number of equal particles are fastened at unequal intervals to a fine string, and then collected into a heap at the edge of a smooth horizontal table, with the extreme one just hanging over the edge. The intervals are such that the times between successive particles being carried over the edge are equal: prove that if c_n be the interval between the n^{th} and the $(n+1)^{th}$ particle, and v_n the velocity just after the $(n+1)^{th}$ particle is carried over, then $\frac{c_n}{c_1}=\frac{v_n}{v_1}=n$.

Professor Wolstenholme, *Educ. Times.*

If v be the velocity acquired by the first particle during its fall through the interval c_1, we get immediately, from the conditions of the problem, the two series of relations,

$$v_1=\tfrac{1}{2}v,\quad v_2=\tfrac{2}{3}(v_1+v)=\tfrac{2}{2}v,\quad v_3=\tfrac{3}{4}(v_2+v)=\tfrac{3}{2}v,\quad \text{\&c.}$$

$$2gc_1=v^3,\quad 2gc_2=(v_1+v)^2-v_1^2=2v^2,\quad 2gc_3=(v_2+v)^2-v_2^2=3v^2,\quad \text{\&c.}$$

Hence

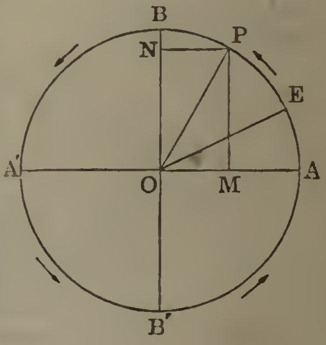
$$v_1:v_2:v_3:\text{\&c.}:v_n=c_1:c_2:c_3:\text{\&c.}:c_n=1:2:3:\text{\&c.}:n.$$

CHAPTER V.

CIRCULAR MOTION.

SECTION I.—*Harmonic Motion.*

85. **Uniform Circular Motion.**—If a point P describe a circle with a uniform motion, the radius of the circle is called the *amplitude* of the motion, and the time of making one revolution is called its *period*. If the arcs are measured from a fixed point A, and the time counted from the instant the moving point passed through a fixed point E, then the angle AOE is called the angle of epoch, or briefly, *the epoch*. Also the ratio which the arc PE, at any instant, bears to the circumference of the circle is called the *phase* of the moving point at that instant.



The arrowheads on the figure denote the direction in which the motion is supposed to take place, and such a rotation as there represented, *i.e.* in the opposite direction to that of the hands of a clock, is considered a positive rotation: that in the opposite direction, or *clockwise*, being considered negative.

Let ω be the angular velocity of P, or the circular measure of the arc described in one second, ε the circular measure of the epoch AOE, and θ that of AOP, we have $\theta = \omega t + \varepsilon$.

Again, if T denote the period, we get $\omega = \frac{2\pi}{T}$, and hence, if desirable, we should write

$$\theta = \frac{2\pi}{T} t + \varepsilon ; \qquad (1)$$

but the form $\theta = \omega t + \varepsilon$, being more compendious, is that we shall generally employ.

86. **Harmonic Motion.**—If PM be drawn perpendicular to the diameter AA', then as P moves uniformly round the circle, the point M moves backwards and forwards along the line AA', and is said to have a *simple harmonic motion.* The amplitude, period, epoch, and phase of the harmonic motion are the same as those of the corresponding circular motion.

If $OM = x$, then the position of M at any instant is given by the equation

$$x = a\cos(\omega t + \varepsilon), \tag{2}$$

where a represents the amplitude, and ε the epoch of the motion. The angle $\omega t + \varepsilon$ is called the *argument* of the motion, and the distance x is said to be a *simple harmonic function of the time.*

Again, if PN be perpendicular to OB, and $y = ON$, we have

$$y = a\sin(\omega t + \varepsilon) = a\cos(\omega t + \varepsilon - \tfrac{1}{2}\pi).$$

Hence the point N has also a harmonic motion, and we infer that a uniform circular motion is equivalent to two simultaneous rectangular harmonic motions, of the same amplitude and period, but differing one-fourth in phase: and conversely.

Again, if the point M be projected on any line, the projected point plainly has a harmonic motion of the same period and phase, but having for amplitude the projection of the amplitude of M.

If we differentiate equation (2) we get

$$v = \frac{dx}{dt} = -a\omega\sin(\omega t + \varepsilon).$$

Consequently the velocity of a point which has a simple harmonic motion is a simple harmonic function of the time, and its maximum value is equal to the velocity in the circle.

Again, the acceleration f is given by the equation

$$f = \frac{dv}{dt} = -\omega^2 a\cos(\omega t + \varepsilon) = -\omega^2 x.$$

Consequently the acceleration at any instant is proportional to the distance from the middle point of the motion, and is always directed towards that point. The acceleration at either extremity of the motion is $-\omega^2 a$.

Any number of harmonic motions of equal periods in the same line are equivalent to a single harmonic motion.

For let $x = a\cos(\omega t + \varepsilon) + a'\cos(\omega t + \varepsilon') + \&c.$

Then $$x = A\cos\omega t - B\sin\omega t,$$

where $$A = \Sigma a\cos\varepsilon, \quad \text{and} \quad B = \Sigma a\sin\varepsilon.$$

Hence $x = C\cos(\omega t + \gamma)$, where

$$C = \sqrt{A^2 + B^2}, \text{ and } \tan\gamma = \frac{B}{A}.$$

This result admits also of a simple geometrical demonstration.

87. **Elliptic Harmonic Motion.**—If a circle be projected orthogonally on any plane its projection is an ellipse, and the projection of any point which moves uniformly on the circle is said to have an *elliptic harmonic motion.*

An elliptic harmonic motion may be resolved into two simple harmonic motions of the same period, along any two conjugate diameters of the ellipse, these motions differing one-fourth in phase. This follows immediately from the property that rectangular diameters in the circle are projected into conjugate diameters in the ellipse. Conversely, any two simple harmonic motions, in different lines, of the same period and differing one-fourth in phase, compound an elliptic harmonic motion, having the lines for conjugate diameters.

EXAMPLES.

1. If two harmonic motions in the same line have equal amplitude (a) and equal periods, but different epochs, ϵ, ϵ', find the amplitude of their resultant.

Ans. $2a\cos\frac{1}{2}(\epsilon - \epsilon')$

2. If the difference of phase in the last passes continuously from 0 to 2π, find the mean value of the square of the amplitude of the resulting vibration.

Ans. $2a^2$.

The mean value is represented by the definite integral (*Int. Calc.*, Art. 221).

$$\frac{8a^2}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\phi\, d\phi.$$

This result is of importance in the Wave Theory of Light, as it shows that the intensity of light is proportional to the square of the amplitude of the vibration which constitutes the light.

3. If two or more harmonic motions in different directions have the same periods and phases, show that their resultant is also a simple harmonic motion of the same phase.

4. Prove that the resultant of any number of simple harmonic motions, differing in directions and phases, but having the same period, is an elliptic harmonic motion.

5. In elliptic harmonic motion prove that the areal velocity of the moving point, round the centre, is constant.

6. Prove that any simple harmonic motion is equivalent to two circular vibrations, in opposite directions.

7. A horizontal shelf moves vertically with simple harmonic motion, the complete period being one second. Find the greatest amplitude in centimetres that objects resting on the shelf may remain in contact with it when at its highest point: assuming $g = 981$. *Ans.* 24·85.

8. In elliptic harmonic motion, being given the difference of phase and the ratio of the amplitudes of the components along two given right lines, perpendicular to each other, determine the position and the ratio of the axes of the ellipse.

If k be the ratio of amplitudes, $\frac{\epsilon}{2\pi}$ the difference of phase, μ the ratio of axes, and α the angle made by the axis major with the direction of greater amplitude, then

$$\mu^2 = \frac{1 + k^2 - \sqrt{1 + 2k^2 \cos 2\epsilon + k^4}}{1 + k^2 + \sqrt{1 + 2k^2 \cos 2\epsilon + k^4}}, \quad \tan 2\alpha = \frac{2k \cos \epsilon}{1 - k^2}.$$

9. Show that two simple harmonic motions, in rectangular directions, of the same epoch, and whose periods are as 1 : 2, compound a parabolic vibration. In this case the motion may be represented by the equations

$$x = a \cos 2\omega t, \quad y = b \cos \omega t.$$

Hence, eliminating t, we get $\frac{x}{a} = \frac{2y^2}{b^2} - 1.$

10. In the same case, if the vibrations differ in epoch, show that the harmonic motions compound a curve of the fourth degree. The motion is represented by the equations

$$x = a \cos (2\omega t - \epsilon), \quad y = b \cos \omega t.$$

Hence, eliminating t, we get

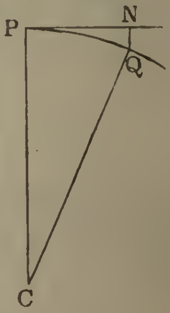
$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) + \frac{2x}{a}\left(1 - \frac{2y^2}{b^2}\right) \cos \epsilon + \cos^2 \epsilon = 0.$$

SECTION II.—*Centrifugal Force.*

88. **Centrifugal and Centripetal Force.**—A heavy particle may be made to move in a circle, either by having it attached to a fixed point by an inextensible string, and made to move on a plane passing through the fixed point, or by its being constrained to move in a fixed circular groove.

During the motion in the former case the string sustains a strain or tension: in the latter, the moving particle presses outwardly against the groove. This tension, or pressure, is called the centrifugal force of the particle, and is always directed outwards from the centre of the circle described.

The groove, or string, exerts at the same instant an equal and opposite reaction, inwards on the particle. This latter is called the centripetal force which acts on the particle. If m be the mass of the particle, and V its velocity at any instant, then, by Art. 25, we infer that the centripetal force is represented by $m\frac{V^2}{r}$, where r is the radius of the circle.



This result can also be established otherwise in the following manner. Let P be the position of the particle at any instant, then if it were acted on by no force it would move along the tangent at P with the velocity V, which it has at the instant, and at the end of the time Δt would arrive at the point N, where $PN = V \times \Delta t$: accordingly QN denotes the space through which it has moved, in the time Δt, owing to the centripetal force. This force is directed towards the centre of the circle, and may be regarded as constant in magnitude, and direction, during the indefinitely short time Δt.

Hence, if f denote its acceleration, we have, by Art. 35,

$$QN = \tfrac{1}{2} f (\Delta t)^2.$$

But, in the limit, $PN^2 = 2QN.PC$,

where C is the centre of the circle;

$$\therefore \quad V^2(\Delta t)^2 = 2QN.PC.$$

Hence $$f = \frac{V^2}{r}. \qquad (1)$$

Or, *the centrifugal acceleration f is a third proportional to the radius of the circle and the velocity of the particle.*

The centrifugal force is accordingly represented by $\frac{mV^2}{r}$.

If it be required to calculate the pressure in pounds due to the centrifugal force we substitute $\frac{W}{g}$ for m, and the preceding expression becomes $\frac{W}{g}\frac{V^2}{r}$.

Since the centripetal force is always directed to the centre of the circle, and consequently at right angles to the direction of motion, it has no effect in altering the velocity of the moving particle. Hence if no other force act on the particle its velocity will be constant during the motion.

Conversely, if a particle m describe a circle of radius r, with a uniform velocity V, we infer that the resultant of all the forces which act on it passes through the centre of the circle, and is represented by $\frac{mV^2}{r}$.

Again, as the velocity in the circle is uniform, if T denote the number of seconds in which the circumference is described, we have $V = \frac{2\pi r}{T}$.

Hence, in this case, we have

$$f = 4\pi^2 \frac{r}{T^2}. \qquad (2)$$

Consequently, in uniform circular motion, the centrifugal force varies directly as the radius of the circle, and inversely as the square of the time of revolution.

Again, if ω be the angular velocity of the radius CP, we have $\omega = \frac{2\pi}{T}$: accordingly, in terms of the angular velocity and the distance, we have

$$f = \omega^2 r. \qquad (3)$$

EXAMPLES.

1. Calculate the centripetal acceleration of a particle which moves in a circle of 5 feet radius with a velocity of 10 feet per second. *Ans.* 20.

2. A particle performs 20 revolutions per minute in a circle of 1 foot circumference: find its centrifugal acceleration. *Ans.* ·6981.

3. A body of 1 lb. mass revolves, in a horizontal plane, at the extremity of a string 10 feet long, so as to make a complete revolution in 2 seconds; find the tension of the string in pounds. *Ans.* $\frac{10\pi^2}{g}$.

4. A railway carriage of 1 ton weight is moving at the rate of 60 miles an hour round a curve of 1210 feet radius; find the centrifugal pressure on the rails. *Ans.* 448 lbs.

5. In the last example, find how much the outer rail should be raised in order that the total pressure should be equally distributed between both the rails, the distance between the rails being four feet. *Ans.* $9\frac{1}{2}$ inches, approximately.

89. **Circular Orbits.**—In the case of uniform circular motion, since the centrifugal force acting on the particle at each instant is directed from the centre of the circle, we may suppose the particle to be kept in its circular orbit by the action of an attractive force always directed to the centre of the circle, and whose acceleration is $4\pi^2 \frac{r}{T^2}$. Hence, if the magnitude of the acceleration directed to a fixed centre of force be known, we can determine the conditions that a particle should describe a circle, having the fixed point as its centre. For, if f be the attraction caused by the central force at the distance r of the particle, we must have $f = \frac{v^2}{r}$, and therefore $v = \sqrt{fr}$. This determines the velocity at each point in the circle.

Conversely, if the particle be projected at the distance r from the centre of force, at right angles to the radius vector, with a velocity $v = \sqrt{fr}$, it will proceed to describe a circle freely round the centre of force.

Also, the time T of describing the circle will be $\frac{2\pi r}{v}$ or $2\pi\sqrt{\frac{r}{f}}$.

For example, if the attractive force be directly proportional to the distance, we have $f = \mu r$, where μ is some constant; and consequently, in this case,

$$T = \frac{2\pi}{\sqrt{\mu}}. \qquad (4)$$

Hence, for this law of attraction, the time of revolution in a circular orbit is independent of the distance; and we infer that the times of revolution for all circular orbits round the same centre of force are equal.

Again, let the attraction vary inversely as the square of the distance from the centre, that is to say, let $f = \frac{\mu}{r^2}$.

In this case the velocity in the circular orbit is represented by $\sqrt{\frac{\mu}{r}}$, and the time of revolution by $2\pi\sqrt{\frac{r^3}{\mu}}$. Hence we see that in different circular orbits round the same centre of force (which varies as the inverse square of the distance), the *squares of the periodic times vary as the cubes of the distances from the centre of force.*

This establishes Kepler's Third Law for circular orbits.

The preceding are particular cases of general results, connected with the problem of Central Forces.

90. **Centrifugal Force at Earth's Equator.**—We now proceed to consider the centrifugal force arising from the rotation of the Earth on its axis.

Let r be the number of feet in the Earth's radius; T the number of seconds in the time of a complete rotation on its axis; f the acceleration due to centrifugal force at the Equator, and we have

$$f = 4\pi^2 \frac{r}{T^2}.$$

The most convenient method of determining f is by comparing its value with that of g at the Equator: thus

$$\frac{f}{g} = \frac{4\pi^2 r}{gT^2}. \qquad (5)$$

Substituting their numerical values for π, r, g, and T, we find, to the nearest integer,

$$g = 288f.$$

Again, as the centrifugal force, in this case, tends to diminish the action of gravity, it should be added to the observed value of g, to obtain the true acceleration due to the Earth's attraction at the Equator.

Hence we have $G = g + f = 289f$,

$$\therefore \quad f = \frac{G}{289}, \tag{6}$$

or, the centrifugal force at the Equator is the 289th part of the Earth's attraction at the same place, approximately.

It is easy from this result to determine what the time of the Earth's rotation should be in order that bodies should have no weight at the Equator.

For let T' be the time required, then the corresponding centrifugal acceleration would be $\frac{4\pi^2 r}{T'^2}$;

hence
$$\frac{4\pi^2 r}{T'^2} = G = 289 \, . \, \frac{4\pi^2 r}{T^2};$$

$$\therefore \; T' = \frac{T}{17}. \tag{7}$$

Accordingly, if the Earth were to rotate 17 times faster than it does, bodies would lose all their weight at the Equator.

Hence also we infer that if a body revolve around the Earth in a circle, near its surface, and subject to its attraction solely, it should travel round the circumference of its circular orbit in the 17th part of a day.

91. **Verification of the Law of Attraction.**—The last result can be applied to verify, by a rough calculation, the fact that the Moon is retained in her orbit by the attractive force of the Earth; the law of force being the inverse square of the distance from the Earth's centre; and the Moon's path being assumed to be a circle having the centre of the Earth as its centre.

For let R denote the distance of the centre of the Moon from the Earth's centre; T her periodic time expressed in days; T' that of a body revolving round the Earth close to its surface.

Then, by Art. 89, we have

$$T^2 : T'^2 = R^3 : r^3. \qquad (8)$$

Assuming the Moon's distance from the Earth's centre to be 60 times the Earth's radius, as found approximately by observation, we have $R = 60\,r$.

Hence $$T = T'\,60^{\frac{3}{2}} = 60\,T'\sqrt{60}\,;$$

but, by the last Article, $T' = \dfrac{1}{17}$ (since the times T, T' are expressed in days).

Hence $$T = \frac{60}{17}\sqrt{60}.$$

This, when calculated, gives approximately

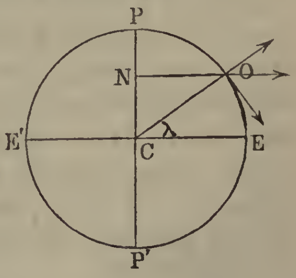
$$T = 27{\cdot}3387 \text{ days.}$$

The near agreement (within 24 minutes) of this result with the mean value of T as obtained by observation, viz., 27·3216 days, affords a strong confirmation of the Law of Gravitation. When more accurate values are substituted, and all the circumstances of the problem taken into account, the calculated agrees completely with the observed result.

EXAMPLE.

Show, roughly, that if a tower were built near the Equator so high that a body at the top should appear weightless, the height of the tower would be less than $\frac{1}{9}$th and greater than $\frac{1}{10}$th of the Moon's distance from the Earth.—*Camb. Trip.*

92. **Tangential and Normal Components.**—Let O represent any place, of latitude λ, on the Earth's surface, supposed spherical: ON the perpendicular drawn from O to the Earth's axis, PP'.



Then the centrifugal acceleration at O is in the direction NO produced; and its amount is represented by

$$\frac{4\pi^2 . NO}{T^2}, \text{ or } \frac{4\pi^2 r \cos\lambda}{T^2} = f\cos\lambda,$$

where f denotes the centrifugal acceleration at the Equator.

The centrifugal force along NO can be resolved into two components; one along CO produced, the other in the tangential direction.

These are plainly represented by $f\cos^2\lambda$, and $f\cos\lambda\sin\lambda$, respectively; or by $\dfrac{G\cos^2\lambda}{289}$ and $\dfrac{G\cos\lambda\sin\lambda}{289}$. The effect of the former is to diminish the Earth's attraction, as before.

Hence for the actual value of g at any latitude λ, assuming the Earth an exact sphere, we get

$$g = G - \frac{G\cos^2\lambda}{289} = G\left(1 - \frac{\cos^2\lambda}{289}\right). \qquad (9)$$

This result has to be modified when the spheroidal form of the Earth is taken into account.

The tangential component of the centrifugal force vanishes at the Equator and also at the poles. For intermediate places it varies as $\sin 2\lambda$, and has its greatest value at 45° latitude, where it is equal to half the centrifugal acceleration at the Equator.

EXAMPLES.

1. Calculate the diminution of gravity due to centrifugal force at a latitude of 45°.

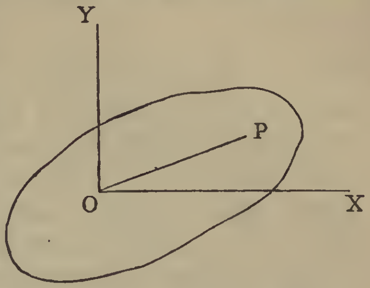
2. Calculate the tangential component for the same case.

3. Calculate the centrifugal acceleration at the Equator of the planet Mercury, its radius being 1570 miles, and time of revolution $24^{h}\ 5^{m}$. *Ans.* ·0435.

93. **Rotation of a Rigid Body.**—If a rigid body be conceived to turn round a fixed axis, each of its points will describe a circle, having its centre on the axis of revolution. Also, since every line in the body that is perpendicular to the axis turns through the same angle, the angular velocity of each point of the rigid body will be the same at any instant. This instantaneous angular velocity is called the *angular velocity of the body*, and it is plainly the same as the velocity of any point in the body which is at the unit of distance from the fixed axis.

If the angular velocity at any instant be represented by ω, the velocity of a point whose distance from the axis is p is represented by $p\omega$.

94. *If a plane lamina rotate about a fixed axis, at right angles to its plane, the centrifugal forces of the different elements of the lamina are equivalent to a single force, passing through its centre of mass, and which is the same as if the entire mass were concentrated at that point.*



Let the plane of the paper represent that of the lamina; and take the point O, in which the fixed axis meets the plane, as the origin of a pair of *fixed* rectangular axes OX and OY.

Suppose ω to be the angular velocity of the lamina at any instant; then, since each point in the lamina describes a circle round O, the centrifugal forces for all its elements pass through that point; they accordingly are equivalent to a single force. To find the value of this single resultant; let $OP = r$, and let dm denote the mass of an element at the point P; then the centrifugal force of the element is $\omega^2 r\,dm$, acting along the line OP produced. This force can be decomposed into two, $\omega^2 x\,dm$ and $\omega^2 y\,dm$, parallel to the axes of x and y respectively.

Suppose the centrifugal forces of the other elements resolved in like manner, then the entire system is equivalent to the forces $\omega^2 \Sigma x\,dm$ and $\omega^2 \Sigma y\,dm$, parallel to OX and OY. But

$$\Sigma x dm = M\bar{x}, \quad \Sigma y dm = M\bar{y},$$

where $\bar{x}$, $\bar{y}$ are the coordinates of the centre of mass of the lamina.

Hence the resultant of the entire system of centrifugal forces is the same as that of the two forces

$$\omega^2 M\bar{x} \quad \text{and} \quad \omega^2 M\bar{y};$$

or to the single force $\omega^2 Md$, where d denotes the distance of the centre of mass of the lamina from the fixed axis.

95. A similar theorem holds for any uniform rigid body turning round a fixed axis, provided the body has a *plane of symmetry* passing through the axis.

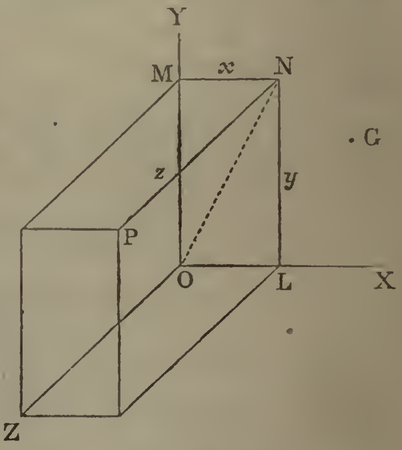
For the body may be conceived divided into a number of indefinitely thin parallel plates, by planes perpendicular to the fixed axis, and the preceding theorem holds for each of these plates or laminæ.

Again, if m_1, m_2, m_3, . . . &c., denote the masses of the plates; and p_1, p_2, p_3, . . . &c., the distances of their respective centres of gravity from the fixed axis; then as the body is supposed uniform and symmetrical, the centres of mass of each of the plates all lie in the plane of symmetry; therefore the forces $\omega^2 m_1 p_1$, $\omega^2 m_2 p_2$, . . . form a system of parallel forces. They consequently have a single resultant, equal to their sum, or to $\omega^2 Md$, where M denotes the mass of the body, and d the distance of its centre of mass from the axis of rotation.

Hence, the *stress* on the fixed axis produced by the centrifugal forces is in this case, in general, represented by $\omega^2 Md$.

If the fixed axis pass through the centre of mass the stress on the axis becomes a momental stress or a couple: as will be shown also in the following Article.

96. Next, let us consider the case of any rigid body rotating round a fixed axis. Suppose a plane drawn through the centre of mass, perpendicular to the fixed axis, and meeting it in the point O. Take O as the origin, the fixed axis as that of z, and two rectangular axes as those of x and y, respectively. Let dm denote an element of mass at the point P, whose coordinates are x, y, z; then, by Art. 94, the centrifugal force for the element dm is equivalent to the forces, $\omega^2 x dm$ and $\omega^2 y dm$, acting at P, parallel to OX and OY, respectively.



Transferring these forces to the point N, they are equivalent to the forces $\omega^2 x dm$, $\omega^2 y dm$, parallel to OX and OY; and to the couples $\omega^2 xz dm$, $\omega^2 yz dm$, parallel to the planes XZ and YZ, respectively. The resultant of the forces $\omega^2 x dm$, $\omega^2 y dm$, acting at N, is obviously directed to O; and consequently if it be transferred to O, it can be resolved into $\omega^2 x dm$ and $\omega^2 y dm$, acting along OX and OY, respectively.

If each of the centrifugal forces be resolved in like manner, the whole system is equivalent to the forces

$$\omega^2 \Sigma x\,dm \quad \text{and} \quad \omega^2 \Sigma y\,dm,$$

or to

$$\omega^2 M\bar{x} \quad \text{and} \quad \omega^2 M\bar{y},$$

acting along OX and OY, respectively; and to the couples $\omega^2 \Sigma xz\,dm$, $\omega^2 \Sigma yz\,dm$, acting in the planes of XZ and YZ, respectively.

If the fixed axis be a principal axis relative to the point O (*Int. Calc.*, Art. 214), we have

$$\Sigma xz\,dm = 0, \quad \text{and} \quad \Sigma yz\,dm = 0.$$

Hence, in this case the strain on the axis produced by the rotation is the same as if the entire mass was concentrated at the centre of gravity of the rigid body.

If, further, the fixed axis be one of the principal axes passing through the centre of gravity of the body, the centrifugal forces arising from the rotation produce no strain on the fixed axis. And accordingly, if, from any cause, a rigid body commence to rotate about such an axis, it will continue to rotate permanently round the axis, provided the only external force be that of gravity.

For example, if we suppose a homogeneous sphere, whose centre is fixed, to receive any impulse, it will commence to rotate around some one of its diameters; and, as every diameter is in this case a principal axis, it follows, from the preceding, that it will continue to revolve permanently round that axis, if we suppose no external force but gravity to act on it.

On account of the property established above, it is of importance, in order that any machine should work smoothly, that the centre of gravity of any wheel, or portion which rotates rapidly, should lie on the axis of rotation, which should be a principal axis; for otherwise the centrifugal forces would cause strong disturbing vibrations.

The theorems of this section are particular cases of important general results which shall be discussed subsequently.

EXAMPLES.

1. A string of 5 feet length is just capable of supporting a weight of 10 lbs.; find the greatest number of revolutions per minute that a weight of 4 lbs. attached to the extremity of the string is capable of making in a horizontal plane without breaking the string. *Ans.* 38.

2. A mass of 8 lbs. is suspended from the extremity of a string 10 feet long; find the least velocity that should be given to it in order to break the string, if its breaking tension be 12 lbs. *Ans.* 12·64 feet per second.

3. Two balls weighing 6 lbs. each are fixed at the extremities of a rod of 10 feet length, which revolves 100 times in a minute around a central vertical axis; find the tension of the connecting rod. *Ans.* 102 lbs.

4. If two equal bodies moving on a rough horizontal plane be connected by a string of invariable length a, but without weight; find the longest time that one can continue to move after the other has been stopped by friction. *Ans.* $\sqrt{\dfrac{a}{\mu g}}$.

5. A body m sliding on a perfectly smooth horizontal table is connected by a string passing through a smooth hole in the table, with another body m' which hangs freely; find the condition that m' should remain at rest, and also the time of revolution of m in its circular path, supposed of radius a.

Ans. Velocity of m should be $\sqrt{\dfrac{m'ga}{m}}$: time of revolution $= 2\pi\sqrt{\dfrac{ma}{m'g}}$.

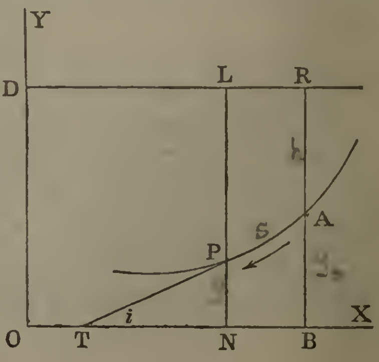
6. If a body, attached at its centre of mass to one end of a string of length r, the other end of which is attached to a fixed point on a smooth horizontal plane, makes n revolutions per second; prove that the tension of the string is to the pressure on the plane as $4\pi^2 n^2 r$ to g.

Prove that at the Equator a shot fired westward with velocity 8333, or eastward, with velocity 7407 metres per second, will, if unresisted, move horizontally round the Earth in one hour and twenty minutes, and one hour and a-half respectively. *Camb. Trip.*, 1878.

7. A rigid body of any form revolves freely round an axis in space: required the conditions under which the centrifugal forces of its several elements will have—(*a*) no resultant; (*b*) a resultant pair; (*c*) a resultant single force; (*d*) a resultant pair and single force. *Lloyd Exhib.*, 1872.

SECTION III.—*Motion in a Vertical Circle.*

97. **Velocity in a Smooth Vertical Curve.**—Before the discussion of motion in a circle we shall consider some properties of the motion of a particle, under the action of gravity, on any vertical curve.



Take OX a horizontal, and OY a vertical line in the plane as axes of coordinates; and suppose x, y the coordinates of P, the position of the particle at the end of any time t. Let A be its position when $t = 0$, and let $AP = s$; then, by Art. 41, the acceleration along the curve at P is represented by

$$g \sin i = -g\frac{dy}{ds}.$$

Hence we have
$$\frac{d^2s}{dt^2} = -g\frac{dy}{ds}. \tag{1}$$

Multiply by $2\,ds$, and integrate; then

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -2gy + \text{const.}$$

Let $y_0 = AB$, and v_0 = velocity at A, then

$$v_0^2 = -2gy_0 + \text{const.};$$

therefore
$$v^2 - v_0^2 = 2g(y_0 - y). \tag{2}$$

Again, if AR be measured upwards $= h$, the height due to the velocity v_0, and RD be drawn parallel to the axis of x; then

$$v^2 = 2g(h + y_0 - y) = 2gPL. \tag{3}$$

Consequently *the velocity at any point P is the same as that acquired in falling from the horizontal line DR.*

This is an extension of the result given in Art. 47, and is itself a case of the general principle of work which shall be given in the next chapter (*see* Art. 128).

98. **Motion in a Vertical Circle.**—If a particle be constrained to move in a vertical circle under the action of gravity, its velocity at any point, by (2), is the *velocity due to* falling through a certain *height* from a certain horizontal line, or *level.* The motion will be one of complete revolution if this right line lies altogether outside the circle. If the line cut the circle the motion will be oscillatory. We proceed to consider the latter case in the first instance. In this case we may either consider the particle as moving in a smooth circular tube, or as attached by an inextensible string to a fixed point in the centre of the circle, the weight of the string being neglected.

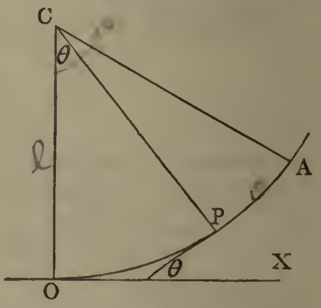
When the arc in which the oscillation has place is but a small portion of the circumference we get what is called a simple pendulum. From this statement the student will see that a *simple pendulum* can only be approximately represented. However, a small leaden ball suspended from a fixed point by a very fine wire may be regarded approximately as a simple pendulum.

99. **Simple Pendulum.**—Let C be the centre of the circle; O its lowest point; A the point from which the particle may be supposed to start; P its position at the end of any time t; v the corresponding velocity;

$$\theta = \angle PCO, \quad \alpha = \angle ACO,$$

estimated in circular measure,

$$s = AP, \quad l = OC.$$



Then, since the velocity at P is that due to falling from a horizontal line drawn through A, we have

$$v^2 = 2gl(\cos\theta - \cos\alpha);$$

but
$$v^2 = \left(\frac{ds}{dt}\right)^2 = l^2\left(\frac{d\theta}{dt}\right)^2;$$

therefore
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$= \frac{4g}{l}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right).$$

Consequently
$$\frac{d\theta}{dt} = \pm 2\sqrt{\frac{g}{l}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}.$$

Again, since in the motion from A to O, θ diminishes as t increases, $\frac{d\theta}{dt}$ is negative. Accordingly we have

$$\frac{d\theta}{dt} = -2\sqrt{\frac{g}{l}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}.$$

Hence we get
$$t\sqrt{\frac{g}{l}} = -\int_{a}^{\theta} \frac{d\theta}{2\sqrt{\sin^2\frac{a}{2} - \sin^2\frac{\theta}{2}}}. \tag{4}$$

100. **Time of a Small Oscillation.**—The preceding definite integral which represents the time of describing a circular arc cannot be expressed in finite terms by means of the ordinary algebraic or trigonometrical functions; however when the amplitude of the oscillation is small we can easily get an approximate value for t, as follows:—When a is so small, that we may neglect powers of a and θ beyond the second, we have

$$4\left(\sin^2\frac{a}{2} - \sin^2\frac{\theta}{2}\right) = a^2 - \theta^2.$$

Hence (4) becomes,

$$\sqrt{\frac{g}{l}}\, t = -\int_{a}^{\theta} \frac{d\theta}{\sqrt{a^2 - \theta^2}} = \cos^{-1}\left(\frac{\theta}{a}\right).$$

No constant is added as $\theta = a$ when $t = 0$. Consequently we have

$$\theta = a \cos\sqrt{\frac{g}{l}}\, t. \tag{5}$$

Accordingly θ is a simple harmonic function of the time (Art. 86).

Again, when $\theta = 0$, we have $\sqrt{\frac{g}{l}}\, t = \frac{\pi}{2}$; hence the time of descent to the lowest point is represented by $\frac{\pi}{2}\sqrt{\frac{l}{g}}$.

The particle, after arriving at the lowest point, plainly moves up the other side of the arc, and if the whole time of a small oscillation, *expressed in seconds*, be denoted by T, we have

$$T = \pi\sqrt{\frac{l}{g}}. \tag{6}$$

Since this expression is independent of a, it follows that the *time of a small oscillation is the same for all arcs of vibration in the same circle.* From this property the vibrations of a pendulum are said to be *isochronous.* Also the time of a small oscillation at any place varies as the square root of the length of the pendulum.

101. **The Seconds Pendulum.**—A pendulum which oscillates once in every second is called a seconds pendulum. If L be its length, since the corresponding value of T is unity, we have

$$g = \pi^2 L. \tag{7}$$

Hence the value of g can be determined for any place whenever the corresponding value of L is obtained.

This gives the most accurate method of finding the value of g at any place, since that of L can be determined with great accuracy by observation.

Any rigid body made to vibrate about a fixed horizontal axis is called a *compound pendulum.* It will be shown subsequently (*see* Art. 133) that in every such case there is an *equivalent simple* pendulum which would vibrate in the *same time* as the actual pendulum under consideration. This circumstance renders the consideration of the *ideal* pendulum above discussed of the utmost practical importance.

The length of a seconds pendulum at London is found to be 39·1416 inches, approximately, and hence the corresponding value of g is 32·1908 feet.

Pendulum observations furnish the most accurate proof of the fact that the force of gravity acts with equal intensity on all substances; as it will be seen that the length of the simple pendulum equivalent to any compound one depends merely on the shape of the latter, but not on its material, provided it be homogeneous.

Again, if T, T' be the times of small oscillations of two pendulums of different lengths, l and l'; and if n and n' be the number of their respective vibrations in the same time (a day suppose), we shall have

$$\frac{n}{n'} = \frac{T'}{T} = \sqrt{\frac{l'}{l}}. \tag{8}$$

Hence, if the length l of any simple pendulum be known, and also the number n of its vibrations in a day, the length L of the seconds pendulum at the place can be calculated. For, since the number of seconds in a day is 86400, we have, from formula (8),

$$L = \left(\frac{n}{86400}\right)^2 l. \tag{9}$$

The time T of the vibration of a pendulum varies either, (1) by altering the length l of the pendulum, or (2) by changing the place of vibration. We shall consider these causes independently.

102. **Change of Length.**—Adopting the same notation as before, we get

$$\frac{n^2}{n'^2} = \frac{l'}{l};$$

hence
$$\frac{n^2 - n'^2}{n'^2} = \frac{l' - l}{l}; \quad \therefore\ n - n' = \frac{n'^2}{n + n'}\,\frac{l' - l}{l}.$$

When the change in length is a very small fraction of the whole length, n and n' are nearly equal, and we have, approximately.

$$\frac{n'^2}{n + n'} = \frac{n}{2}.$$

Accordingly, in this case,

$$n - n' = \frac{n}{2}\,\frac{\Delta l}{l}; \tag{10}$$

where Δl denotes the change of length of the pendulum.

If the pendulum be lengthened, *i.e.* if Δl be positive, $n - n'$ is positive, and hence the number of vibrations in a given time is diminished when the length of the pendulum is increased, as is also otherwise evident.

In the case of a seconds pendulum we substitute L for l in the preceding; and since $n = 86400$, we get for the diminution in the number of vibrations in a day,

$$43200\,\frac{\Delta L}{L}.$$

Hence we can determine the number of seconds gained or lost by a seconds pendulum in a day when its length is slightly altered.

As bodies in general expand slightly with an increase of temperature, an ordinary clock should go slower in hot weather, and faster in cold. The different methods of compensation for correcting the error arising from this cause will be found in practical treatises on the subject. The amount of expansion for an increase of temperature for different substances has been accurately determined, and registered in Tables.

If ΔL denote the change in the length of a seconds pendulum arising from this cause, the corresponding loss or gain can be determined by (10).

We add a few examples for illustration.

EXAMPLES.

1. Calculate the length of a pendulum beating seconds in London, assuming $g = 32\cdot19$.

2. If the bob of a seconds pendulum be screwed up one turn, the screw being 32 threads to the inch; find the number of seconds it should gain in the day in consequence, assuming $L = 39\cdot14$ inches. *Ans.* 34·7 seconds.

3. A heavy ball, suspended by a fine wire, makes 885 oscillations in an hour. Find the length of the wire approximately, assuming the length of the seconds pendulum to be 39·14 inches. *Ans.* 54 feet.

4. Find the error in one day produced by an increase of 15° F. of temperature in a steel seconds pendulum; assuming that $\frac{\Delta l}{l}$ for 10° F $= \frac{1}{15600}$. *Ans.* 4·15 seconds.

5. A seconds pendulum is lengthened $\frac{1}{10}$th of an inch; find the number of seconds it will lose in one day. *Ans.* 110·4.

103. **Change of Place.**—The acceleration g varies* from place to place, and consequently the number of vibrations in a given time will vary with the place for the same pendulum.

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* For places at the sea level, this arises from two causes—one, the variation of centrifugal force already considered (Art. 93); the other, that the Earth is not an exact sphere, but is more nearly an oblate spheroid of revolution round its axis of rotation. From each of these it arises that the value of g diminishes in proceeding from the pole to the equator. It was from the observation by Richer, in 1672, that a clock lost two minutes daily when taken to Cayenne, lat. 5° N., and that when the corrected pendulum was brought back to Paris it gained an equal amount, that the variation of the force of gravity on the Earth's surface was first established. The explanation is due to Huygens.

Suppose n and n' to represent the number of vibrations made in one day by the same pendulum at two places, at which g and g' are the corresponding accelerations, we have

$$\frac{n'}{n} = \frac{T}{T'} = \sqrt{\frac{g'}{g}}, \quad \text{or} \quad \frac{n'^2}{n^2} = \frac{g'}{g}.$$

Hence, as before, for one and the same pendulum,

$$n' - n = \frac{n}{2}\,\frac{g' - g}{g}. \tag{11}$$

From this, if L and L' be the lengths of the seconds pendulum at the two places, we get

$$n' - n = \frac{n}{2}\,\frac{L' - L}{L}. \tag{12}$$

It is shown by theory, and verified by observation, that the variation in the length of L, and consequently in g, at the sea level, is proportional to the square of the sine of the latitude (compare Art. 92). Thus, if L denote the length of the seconds pendulum at the equator, L' that at latitude λ', we have

$$L' = L + m \sin^2 \lambda'. \tag{13}$$

Hence, if L_1 be the length of the seconds pendulum at 45° latitude, we have $L_1 = L + \frac{m}{2}$.

Eliminating L, we get

$$L' = L_1 - \frac{m}{2} \cos 2\lambda'. \tag{14}$$

Again, if L'' be the length corresponding to the latitude λ'', and g'' the corresponding value of g, we have

$$\begin{aligned} \frac{g' - g''}{g'} &= \frac{m}{2L'}(\cos 2\lambda'' - \cos 2\lambda') \\ &= \frac{m}{L_1} \sin(\lambda' + \lambda'') \sin(\lambda' - \lambda''), \text{ approximately.} \end{aligned}$$

By accurate observation of the number of vibrations lost by a pendulum which beats seconds at the latitude λ, when taken to a latitude λ' the value of $\frac{m}{L_1}$ can be determined.

Such observations give $\frac{m}{L_1} = \frac{1}{195}$, approximately, and $L_1 = 39{\cdot}118$ inches.

Hence, we get,

$$L' = 39{\cdot}118 - \tfrac{1}{10}\cos 2\lambda'.$$

Again, suppose a pendulum, beating seconds at any place, taken to the height h above the Earth's surface at that place; and let g' be the value of g for the new position; then, since the force of gravity varies as the inverse square of the distance from the Earth's centre, we have

$$g' = g\frac{r^2}{(r+h)^2} = g\left(1 - \frac{2h}{r}\right), \text{ approximately,}$$

where r denotes the length of the Earth's radius; therefore

$$\frac{g' - g}{g} = -\frac{2h}{r}.$$

Hence, when $\frac{h}{r}$ is a very small fraction, the number of seconds lost in a day by the seconds pendulum is

$$86400\frac{h}{r}.$$

Suppose, for example, $h = 1$ mile, and $r = 3956$ miles, then the number of seconds lost in a day will be 22, approximately.

In this investigation the attraction on the pendulum of the part of the Earth above the sea level has been neglected.

EXAMPLES.

1. If a pendulum, beating seconds at the foot of a mountain, lose 10 seconds in a day when taken to its summit; find approximately the height of the mountain, assuming the radius of the Earth 4000 miles, and neglecting the attraction of the mountain. *Ans.* 2444 feet.

2. How much would a clock gain at the equator in 24 hours if the length of the day were doubled? *Ans.* $112\frac{1}{2}$ seconds, approximately.

104. **Airy's Investigation of the Mean Density of Earth.**—A series of important pendulum experiments were undertaken by Sir G. B. Airy, in the Harton coal mine, for the purpose of determining the mean density of the Earth. He found that a pendulum beating seconds at the surface gained $2\frac{1}{4}$ seconds a-day when taken to the bottom of the mine, 1260 feet deep. The calculations employed in arriving at this result, and in determining from it the Earth's mean density, are very intricate; they will be found in the Royal Society's *Transactions* for the year 1856.

The following is a method of arriving, approximately, at the result:—

Let g, g' denote the accelerations due to gravity at the surface and at the bottom of the mine; then, by equation (11), we have

$$\frac{g'-g}{g} = \frac{2\frac{1}{4}}{43200} = \frac{1}{19200}.$$

Again, let r and r' denote the distances of the upper and lower stations from the centre of the Earth, supposed spherical.

Suppose a concentric sphere described through the lower station, then the attraction of the Earth at the upper station may be regarded as consisting of two parts—one due to the interior sphere, the other to the *couche* or shell, bounded by the two spheres. Again, if we suppose this shell to be of uniform density, it exercises no attraction on the pendulum at the bottom of the mine (Minchin, *Statics*, p. 419). Hence the part of g due to the attraction of the *inner* sphere is represented by

$$g'\frac{r'^2}{r^2}.$$

If f denote the acceleration at the upper station due to the attraction of the shell, we have

$$g = f + g'\frac{r'^2}{r^2} = f + g\left(1 + \frac{1}{19200}\right)\frac{r'^2}{r^2}.$$

Again, let h represent the depth of the mine, then $r' = r - h$;

and, since $\frac{h}{r}$ is very small, we have $\frac{r'^2}{r^2} = 1 - \frac{2h}{r}$, approximately.

Accordingly we get, from the preceding equation,

$$f = g\left(\frac{2h}{r} - \frac{1}{19200}\right).$$

In order to get another expression for f, let M, V, D denote respectively the mass, volume, and mean density of the Earth; and m, v, ρ the corresponding quantities for the shell. We assume that the Earth and the shell each attract as if their whole mass was concentrated at their common centre; in this case we have, approximately,

$$f = g\frac{m}{M} = g\frac{\rho}{D}\frac{v}{V} = g\frac{\rho}{D}\frac{r^3 - r'^3}{r^3} = 3g\frac{\rho}{D}\frac{h}{r}.$$

Hence
$$\frac{2h}{r} - \frac{1}{19200} = 3\frac{h}{r}\frac{\rho}{D}.$$

Substituting 3956 miles for r, and 1260 feet for h, we get

$$D = 2{\cdot}625\,\rho. \tag{15}$$

The determination of the mean density of the Earth is thus reduced to finding the value of ρ; but this is a question of extreme practical difficulty.

From an accurate examination of the mineral components of the stratum of the Earth in the neighbourhood of the mine, ρ was calculated by Airy to be $2\frac{1}{2}$ times the density of water. This would give the mean density of the Earth about 6·66, assuming that of water as unity.

Professor Haughton calculated 2·059 as the value of ρ, (*Phil. Trans.*, July 1856), adopting as his basis Humboldt's investigations of the mean heights of Continents on the Earth's surface, and Rigault's, on the relative areas of land and water. This would give 5·405 as the value of the mean density of the Earth.

105. **Time of Oscillation in General.**—The amplitude of the vibration has hitherto been considered so small that powers of a higher than the second have been neglected.

We now proceed to find a general expression for the time T of vibration for any amplitude.

From (4), since $\frac{1}{2}T$ represents the time to the lowest point on the circle, we get

$$T = \sqrt{\frac{l}{g}}\int_0^a \frac{d\theta}{\sqrt{\sin^2\frac{a}{2} - \sin^2\frac{\theta}{2}}}. \tag{16}$$

Now, assuming* $\sin\frac{\theta}{2} = \sin\frac{a}{2}\sin\phi$, we get

$$\frac{d\theta}{\sqrt{\sin^2\frac{a}{2} - \sin^2\frac{\theta}{2}}} = \frac{2d\phi}{\cos\frac{\theta}{2}} = \frac{2d\phi}{\sqrt{1 - \sin^2\frac{a}{2}\sin^2\phi}}.$$

Also, when $\theta = 0$, we have $\phi = 0$; and when $\theta = a$, $\phi = \frac{\pi}{2}$.

Consequently $$T = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2\frac{a}{2}\sin^2\phi}}. \tag{17}$$

Again, substitute k^2 for $\sin^2\frac{a}{2}$, then since

$$(1 - k^2\sin^2\phi)^{-\frac{1}{2}} = 1 + \frac{1}{2}k^2\sin^2\phi + \frac{1\,.\,3}{2\,.\,4}k^4\sin^4\phi + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}k^6\sin^6\phi + \&c.,$$

and (*Int. Calc.*, Art. 93),

$$\int_0^{\frac{\pi}{2}} \sin^{2m}\phi\,d\phi = \frac{1\,.\,3\,.\,5\,...\,(2m-1)}{2\,.\,4\,.\,6\,...\quad 2m}\,\frac{\pi}{2},$$

we get

$$T = \pi\sqrt{\frac{l}{g}}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 k^4 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2 k^6 + \&c.\right\}. \tag{18}$$

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* This assumption is obviously a legitimate one, because the value of θ during the motion can never be greater than a.

If h be the vertical height of the point from which the pendulum commences to descend above the lowest point in the circle, we have

$$k^2 = \sin^2\frac{a}{2} = \frac{1-\cos a}{2} = \frac{h}{2l};$$

and the preceding result becomes

$$T = \pi\sqrt{\frac{l}{g}}\left\{1 + \left(\frac{1}{2}\right)^2\frac{h}{2l} + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\left(\frac{h}{2l}\right)^2 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\left(\frac{h}{2l}\right)^3 + \ldots\right\} \quad (19)$$

The first term gives the value already arrived at for a small oscillation, and is independent of the amplitude.

The second approximation, which is the one commonly adopted when the lowest powers of the amplitude are taken into account,

$$T = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{h}{8l}\right). \quad (20)$$

In terms of the semi-amplitude a, it is

$$T = \pi\sqrt{\frac{l}{g}}\left\{1 + \frac{a^2}{16}\right\}, \quad (21)$$

in which a is taken in circular measure.

The general equation (16) is immediately integrable in one case, viz., when the velocity at any point on the circle is that due to a fall from its highest point; for in this case we have $a = \pi$, and therefore $\sin\frac{a}{2} = 1$. Equation (4) becomes in this case

$$\frac{d\theta}{\cos\frac{\theta}{2}} = 2\sqrt{\frac{g}{l}}\,dt;$$

hence we get

$$2\sqrt{\frac{g}{l}}\,t = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \text{const.}$$

It may be observed that in this case the particle would take an infinite time to reach the highest point on the circle.

EXAMPLES.

1. How is the value for the time of vibration of a pendulum to be corrected when the length of the arc of vibration is taken into account?

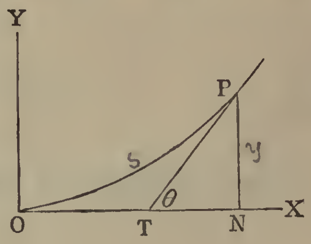
2. Apply to the case where the amplitude of vibration is 120°.

Here $\frac{h}{2l} = \frac{1}{4}$. we have

$$T = \pi \sqrt{\frac{l}{g}} \left(1 + \frac{1}{16} + \frac{9}{1024} + \&c.\right).$$

3. If a pendulum, which beats seconds for very small oscillations, be made to vibrate through an arc of 10°; find, approximately, the number of seconds it should lose in a day. *Ans.* 41.

106. Motion in a Vertical Cycloid.—Let a particle be supposed to move along a smooth cycloid, having its vertex O at its lowest point, and its axis OY vertical.



Calling $OP = s$, $PN = y$, and a = diameter of generating circle. Then (*Diff. Cal.*, Art. 276), we have

$$s^2 = 4ay.$$

Also, from (1),

$$\frac{d^2s}{dt^2} = -g\frac{dy}{ds} = -\frac{g}{2a}s,$$

or

$$\frac{d^2s}{dt^2} + \frac{g}{2a}s = 0. \tag{22}$$

107. Integration of $\frac{d^2x}{dt^2} \pm \mu x = 0$.—As differential equations of the form $\frac{d^2x}{dt^2} \pm \mu x = 0$ are of frequent occurrence in physical problems, we proceed to consider their solution.

There are two cases, according as the upper or lower sign is taken.

1st.—Let
$$\frac{d^2x}{dt^2} + \mu x = 0.$$

Multiplying by $2dx$, and integrating, we get

$$\left(\frac{dx}{dt}\right)^2 + \mu x^2 = \text{const.}$$

To determine the constant, suppose $x = a$ when $\frac{dx}{dt} = 0$, then the constant is plainly μa^2;

therefore
$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2).$$

Hence
$$\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}\, dt,$$

or
$$\sin^{-1}\frac{x}{a} = \sqrt{\mu}\, t + \alpha,$$

where α denotes an arbitrary constant. Consequently,

$$x = a \sin(\sqrt{\mu}\, t + \alpha), \tag{23}$$

where a, α are arbitrary constants, to be determined in each case by the conditions of the problem.

It may be observed that x is a simple harmonic function of the time (Art. 86).

The preceding solution admits of being also written in the form

$$x = C \cos \sqrt{\mu}\, t + C' \sin \sqrt{\mu}\, t, \tag{24}$$

where C and C' are two arbitrary constants. Either of the latter equations may be regarded as the complete integral of the differential equation

$$\frac{d^2x}{dt^2} + \mu x = 0.$$

2nd.— $$\frac{d^2x}{dt^2} = \mu x;$$

proceeding as before, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu(x^2 - a^2);$$

in which a is an arbitrary constant,

or $$\frac{dx}{\sqrt{x^2 - a^2}} = dt\sqrt{\mu};$$

therefore, $t\sqrt{\mu} + \alpha = \int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2})$,

in which α is an arbitrary constant.

Hence $$x + \sqrt{x^2 - a^2} = e^{\alpha + \sqrt{\mu}t} = Ae^{\sqrt{\mu}t},$$

where A is arbitrary.

Again, since

$$(x + \sqrt{x^2 - a^2})(x - \sqrt{x^2 - a^2}) = a^2,$$

we get $$x - \sqrt{x^2 - a^2} = \frac{a^2}{A} e^{-\sqrt{\mu}t}.$$

Adding, we obtain

$$2x = Ae^{\sqrt{\mu}t} + \frac{a^2}{A} e^{-\sqrt{\mu}t},$$

which may be written in the form

$$x = Ce^{t\sqrt{\mu}} + C'e^{-t\sqrt{\mu}}, \qquad (25)$$

when C and C' are two arbitrary constants, to be determined as before, by the conditions of the problem in each particular case.

108. The equation

$$\frac{d^2x}{dt^2} + \mu x + \nu = 0$$

is immediately reducible to the preceding, for it may be written

$$\frac{d^2x}{dt^2} + \mu\left(x + \frac{\nu}{\mu}\right) = 0.$$

If we substitute z for $x + \frac{\nu}{\mu}$, this becomes $\frac{d^2z}{dx^2} + \mu z = 0$; consequently we have

$$x = -\frac{\nu}{\mu} + C \cos t\sqrt{\mu} + C' \sin t\sqrt{\mu}.$$

In like manner the solution of

$$\frac{d^2x}{dt^2} - \mu x + \nu = 0$$

is

$$x = \frac{\nu}{\mu} + Ce^{t\sqrt{\mu}} + C'e^{-t\sqrt{\mu}}.$$

109. **Time of Oscillation in Cycloid.**—Returning to equation (22), Art. 106, substituting s for x, and $\frac{g}{2a}$ for μ, in equation (24), we find for its integral

$$s = c \cos \sqrt{\frac{g}{2a}}\,t + c' \sin \sqrt{\frac{g}{2a}}\,t. \tag{26}$$

In order to determine the constants c and c', suppose the particle to start from rest, at the distance s' from the vertex o (measured along the curve); then we have $s = s'$, and $\frac{ds}{dt} = 0$, when $t = 0$. Making these substitutions in (26), as well as in the equation derived from it by differentiation, we get

$$c = s', \quad \text{and} \quad c' = 0;$$

therefore

$$s = s' \cos \sqrt{\frac{g}{2a}}\,t. \tag{27}$$

Again, when $s = 0$, we get

$$t\sqrt{\frac{g}{2a}} = \frac{\pi}{2}, \text{ or } t = \frac{\pi}{2}\sqrt{\frac{2a}{g}}:$$

this gives the time of descent to the lowest point. If T denote the time of an oscillation, we have

$$T = \pi\sqrt{\frac{2a}{g}}. \tag{28}$$

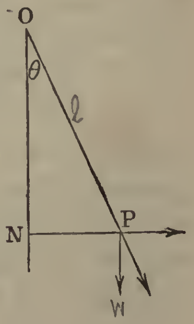
Since this result is independent of the length of the arc of vibration, it follows that the *time of vibration is the same for all arcs of the cycloid*, and the property of *tautochronism*, which in the circle holds only for very small arcs, holds in all cases for the cycloid (compare Art. 86).

The foregoing value of T is the same as that for a small oscillation in a vertical circle of radius $2a$. Moreover, as $2a$ is the radius of curvature at the vertex of the cycloid (*Diff. Calc.*, Art. 276), the duration of an oscillation in a vertical cycloid is the same as that of a small oscillation in the circle which osculates it at its lowest point; as is manifest also from other considerations.

It is readily seen that the time of an indefinitely small oscillation about the lowest point in any plane vertical curve is the same as that in the osculating circle at the lowest point; and its duration is accordingly represented by $\pi\sqrt{\frac{\rho}{g}}$, where ρ denotes the radius of curvature at the point.

110. **Conical Pendulum.**—Suppose the pendulum, instead of moving in a vertical plane, to describe a right cone around a vertical axis, and let P be the position of the revolving particle at any instant; O the point of suspension; PN the perpendicular let fall on the vertical axis;

$$OP = l, \quad \angle PON = \theta.$$



Then the motion of P may be considered as taking place in a horizontal circle, whose centre is N, and radius PN or $l \sin\theta$.

Now in order that this motion should take place it is necessary that the resultant of the tension of the string and the weight of the particle should act along PN, and be equal and opposite to the centrifugal force; *i.e.* that the resultant of the weight W, and the centrifugal force, $\frac{W}{g}\frac{v^2}{l\sin\theta}$, should act in the line OP. This gives

$$W : \frac{W}{g}\frac{v^2}{l\sin\theta} = ON : PN,$$

or

$$v^2 = gl\sin\theta\frac{PN}{ON} = gl\frac{\sin^2\theta}{\cos\theta};$$

therefore

$$v = \sin\theta\sqrt{\frac{gl}{\cos\theta}}. \tag{29}$$

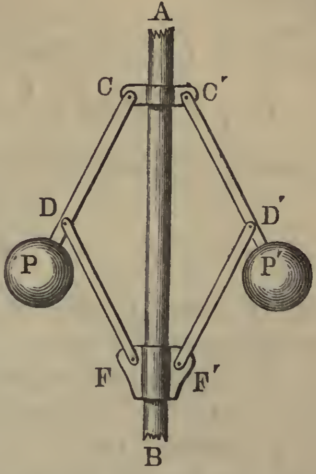
This gives the velocity in terms of θ and l.

Again, if T be the time of revolution, we have

$$T = \frac{2\pi PN}{v} = 2\pi\sqrt{\frac{l\cos\theta}{g}}. \tag{30}$$

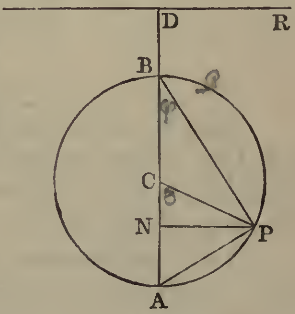
This determines the time of revolution when the angle θ, which the pendulum makes during the motion with the vertical, is known. It is evidently the same as that of a double oscillation in a simple pendulum of length $l\cos\theta$ or ON. The tension of the string is represented by $W\sec\theta$. The preceding is a particular case of the motion of a particle on a smooth sphere, which will be considered subsequently.

111. **Watt's Governor.**—The principle of the conical pendulum was employed by Watt in the instrument called a governor, for the purpose of regulating the supply of steam so as to maintain, approximately, a steady motion in a steam-engine. Its construction, under a form which is commonly employed, is as follows:—



Let *AB* represent a vertical spindle rotating with an angular velocity, whose speed is so regulated as to be always proportional to that of the machine. *CP* and *CP′* are rigid rods, jointed at *C* and *C′* upon the revolving spindle, and having massive equal balls *P* and *P′* fixed at their extremities. *FD* and *F′D′* are two rods also jointed at *D* and *D′* to the rigid rods, and jointed at *F* and *F′* to a *collar*, moveable freely on the spindle. The collar at *F*, sliding freely up and down the spindle, is united to a lever which opens or closes the valve that regulates the supply of steam to the cylinder of the engine. When the shaft *AB* turns too fast, the balls *P* and *P′* fly from it, raising the collar *F*, and thus diminishing the supply of steam, and consequently reducing the speed. For a more complete discussion the student is referred to works on practical mechanics.

112. **Revolution in a Vertical Circle.**—We now return to the question of the revolution of a particle in a vertical circle under the action of gravity.



Suppose *DR* to be the horizontal line to the distance below which the velocity at any point is due, and let $AD = h$; then, by Art. 97, the velocity at any point *P* is given by the equation

$$v^2 = 2g(h - AN) = 2g(h - 2a\sin^2\tfrac{1}{2}\theta),$$

where $PCA = \theta$.

Hence, denoting $\dfrac{2a}{h}$ by k^2, and substituting $a^2\left(\dfrac{d\theta}{dt}\right)^2$ for v^2, we get

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{4g}{ak^2}(1 - k^2\sin^2\tfrac{1}{2}\theta);$$

therefore $$\frac{d\theta}{dt} = -\frac{2}{k}\sqrt{\frac{g}{a}}\sqrt{1 - k^2 \sin^2 \tfrac{1}{2}\theta},$$

in which k is less than unity.

If $\phi = \angle PBA = \frac{\theta}{2}$, the time of describing any arc of the circle is represented by the definite integral

$$k\sqrt{\frac{a}{g}}\int_\beta^\alpha \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad (31)$$

where α and β are the values of ϕ corresponding to the extremities of the arc.

Comparing the result here given with Art. 104, we see that the time of describing any arc of a circle is in this case in a constant ratio to the time of describing a corresponding arc of a second circle, in which the motion is oscillatory.

The time of describing any arc of the circle is, in general, an elliptic function. There is one case, however, in which it admits of a simple expression, viz., where DR is a tangent to the circle, as in Art. 105.

In this case we have $k = 1$, and the definite integral becomes

$$\int_\beta^\alpha \frac{d\phi}{\cos\phi} = \log \frac{\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right)}.$$

The time of motion from any point to the highest point in the circle becomes infinite in this case; accordingly the particle would continually approach the highest point without ever reaching it.

113. **Pressure on Curve.**—If m denote the mass of the particle, then the normal pressure R on the circle consists of two parts, one arising from the centrifugal pressure, the other from the weight; hence we get

$$R = m\frac{v^2}{a} + mg \cos\theta$$
$$= mg\left(\frac{2d}{a} + 3\cos\theta\right), \qquad (32)$$

where $CD = d$.

At the lowest point this becomes $mg\left(\frac{2d}{a} + 3\right)$, and at the highest point $mg\left(\frac{2d}{a} - 3\right)$, and when the string is horizontal, $mg\frac{2d}{a}$.

If the particle, instead of moving in a tube, is attached by a string, of length a, to a fixed point C, and thus constrained to move in a vertical circle, the preceding expression gives the tension of the string for any position. As long as the tension is positive the string remains stretched. At the point where $R = 0$ the tension vanishes, and the particle will leave the circle and proceed to describe a parabola. It is immediately seen that the distance of this point from the line DR is one-third of CD (see fig. of last Article).

These results will be illustrated by the following examples:—

EXAMPLES.

1. A particle slides down the convex side of a vertical circle; determine the point at which it will leave the curve.

Here, since the velocity at the highest point on the circle is zero, we have $d = a$ ~~$d = 0$~~; and accordingly the point at which $R = 0$ is given by the equation $\cos\theta = -\frac{2}{3}$. The geometrical construction is evident.

2. A particle is projected from the lowest point along the inside of a smooth vertical circle; find the least velocity of projection in order that the particle should make a complete revolution in the circle. *Ans.* $\sqrt{5ag}$.

In this case the pressure at the highest point is zero, and at every other point is positive.

3. If the initial velocity be less than that in the preceding example, find the point P at which the particle will leave the circle, and where it will strike it again.

The construction for the point P in question has been given above. After leaving the circle the particle describes a parabola, and the point Q in which it again meets the circle is found by drawing PQ, making with the vertical direction an angle equal to that which the tangent at P makes with the vertical. This result follows immediately from the principle that the vertical circle osculates the parabolic trajectory at P.

4. In the same case find the direction of motion of the particle at the instant it returns to the circle.

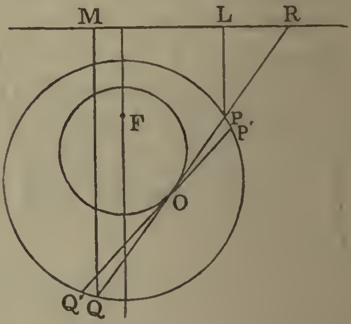
Ans. $\tan\beta = \frac{1}{3}\tan\alpha$, where β is the angle the required direction of motion makes with the vertical, and α is the corresponding angle at the point P, where the particle leaves the circle.

5. Find the velocity of projection from the lowest point on the circle, in order that the particle after leaving the circle should meet it again at its lowest point. *Ans.* $\frac{1}{2}\sqrt{14ga}$.

6. Show that the solution of the general problem of finding the initial velocity, in order that the particle after leaving the circle shall meet it again at a given point, depends on the trisection of an arc.

7. A material particle moves in a circular groove on a smooth inclined plane; if it be projected from its point of rest with a velocity just sufficient to carry it to the highest point in the groove, find the time of its motion.

114. **Lemma on Coaxal Circles.**—A chord PQ of a circle touches a second circle at O; and PL, QM are drawn perpendicular to the radical axis of the two circles: to prove that



$$PO^2 : QO^2 = PL : QM.$$

Let R be the point of intersection of PQ with the radical axis; then, since the tangents from R to the circles are of equal length, we have $RO^2 = RP \cdot RQ$;

therefore $$RQ : RO = RO : RP.$$

Consequently $$QO : OP = RQ : RO,$$

or $$QO^2 : OP^2 = RQ^2 : RQ \cdot RP$$
$$= RQ : RP = QM : PL. \qquad (33)$$

If now we suppose a particle to describe the outer circle with a velocity due to the level LM, and $P'Q'$ be drawn indefinitely near to PQ, touching the inner circle, these tangents may be regarded as intersecting in O, and we accordingly have

$$PP' : QQ' = PO : QO = \sqrt{PL} : \sqrt{QM}.$$

Again, let v, v' be the velocities of the particle when at P and Q respectively; then $v^2 = 2gPL$, $v'^2 = 2gQM$;

therefore $$v : v' = \sqrt{PL} : \sqrt{QM} = PP' : QQ'.$$

Hence the indefinitely small time of describing PP' is the same as that of describing QQ'; consequently the time of motion from P to Q is the same as that from P' to Q', and hence we infer that the time is the same for the description of all arcs cut off by tangents drawn to the inner circle.

EXAMPLES.

1. A particle is moving in a smooth vertical circle under the action of gravity; the time of description of a variable arc of the circle being supposed constant, show that the envelope of its chord is another circle.

2. Show that if the time of motion from P to Q be the same as that from the highest to the lowest point on the circle, the line PQ always passes through a fixed point.

3. Two particles are projected from the same point, in the same direction, and with the same velocity, but at different instants, in a smooth circular tube of small bore, whose plane is vertical. Prove that the line joining them always touches a circle.

4. In the same case, if the particles be projected in opposite directions, the other circumstances being unaltered, prove that the line joining their positions always touches a circle; and find when the circle becomes a fixed point.

5. A particle is moving in a vertical circle under the action of gravity. If three points L, M, N be taken on the circle, find a fourth point P, such that the time of motion from N to P shall be equal to that from L to M.

6. In the same case find P so that the time of describing NP shall be double, or any given multiple of that of describing LM.

7. AB is the vertical diameter of a fine circular tube in which move three equal particles P, Q, Q' (modulus of restitution = 1 for any pair); P starts from A and Q, Q', in opposite senses from B with such velocities that at the first impact all three have equal velocities; prove that throughout the motion the line joining any pair is either horizontal or passes through one of two fixed points, and that the intervals of time between successive impacts are all equal.
Camb. Trip., 1874.

115. Since the time of description, under the action of gravity, of any arc of a vertical circle, is expressible by a definite integral of the form

$$\int_\alpha^\beta \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}},$$

the results of the last Article have important applications in the theory of elliptic functions. For example, they furnish us with simple methods for the addition, subtraction, and multiplication of such functions, depending on elementary properties of coaxal circles. This connexion was first pointed out by Jacobi (*Crelle's Journal*, 1828; *Liouville's Journal*, 1845).

EXAMPLES.

1. Prove that the time of descending any small arc terminated at the lowest point of a vertical circle is to the time down its chord as $\pi : 4$.

2. If the length of a seconds pendulum be 39·14 inches; find the corresponding value of g to two places of decimals.

3. A clock loses 4 minutes in a day; find how much its pendulum should be shortened in order that it may keep correct time. *Ans.* Its $\frac{1}{180}$th part.

4. Assuming the length L in inches of a seconds pendulum at the latitude λ to be given by the formula

$$L = 39{\cdot}118 - \tfrac{1}{10} \cos 2\lambda;$$

find the ratio of the difference between the values of polar and equatorial gravity to equatorial gravity. *Ans.* $\frac{1}{195{\cdot}59}$.

5. Find the correction in the time of vibration of a circular pendulum when the amplitude of the vibration is 30°.

6. If two particles be connected by an inextensible string, and if one be made to move as if under the action of a constant force; prove that the relative motion of the other is that of a simple pendulum.

7. A series of smooth circles in a vertical plane have a common highest point; a particle starting at this point slides down the convex side of each circle; find the locus of the points where the particles leave the circles.

8. A mass m, after sliding down the inner surface of a smooth hemispherical bowl, strikes a mass m' placed at the lowest point of the bowl. If both bodies be perfectly elastic, find the heights to which they respectively ascend after collision.

9. If the length of a conical pendulum be 1 foot, and the weight attached to its extremity be 1 lb.; find approximately the tension of the connecting wire when the time of its revolution is one second. Find also approximately the angle which the revolving wire makes with the vertical spindle.

Ans. Tension $= \dfrac{4\pi^2}{g}$ lb.; $\cos\theta = \dfrac{g}{4\pi^2}$.

10. Investigate the motion of a cycloidal pendulum when acted on by a constant force f, always in a direction opposite to that of its motion, in addition to the force of gravity.

Here the equation of motion is

$$\frac{d^2s}{dt^2} + \frac{g}{2a}s = f,$$

and we get, by Art. 108,

$$s = \frac{2af}{g} + c \cos \sqrt{\frac{g}{2a}}\,t + c' \sin \sqrt{\frac{g}{2a}}\,t.$$

If, when $t = 0$, $s = s'$ and $\frac{ds}{dt} = 0$, we get

$$s = \frac{2af}{g} + \left(s' - \frac{2af}{g}\right) \cos \sqrt{\frac{g}{2a}}\, t.$$

Also
$$\frac{ds}{dt} = - \left(s' - \frac{2af}{g}\right) \sqrt{\frac{g}{2a}} \sin \sqrt{\frac{g}{2a}}\, t.$$

This vanishes when $\sqrt{\frac{g}{2a}}\, t = \pi$; accordingly the time of an oscillation is $\pi \sqrt{\frac{2a}{g}}$; the same as when unresisted.

11. A heavy particle is connected by an inextensible string, 3 feet long, to a fixed point, and describes a circle in a vertical plane about that point, making 600 revolutions per minute; find, approximately, the ratios of the tensions of the string when the particle is at the highest and lowest points, and when the string is horizontal.

12. A body hangs freely from a fixed point by an inextensible string 2 feet in length. It is projected in a horizontal direction with a velocity of 20 feet per second. Compare the tensions at the highest and lowest points of the circle which is described, assuming $g = 32$. *Ans.* 29 : 5.

13. Show that the time of a small oscillation of a pendulum which vibrates in the air is unaffected by its resistance.

The resistance is usually assumed to vary as the square of the velocity. It can accordingly be expressed by a term of the form $\mu \left(\frac{d\theta}{dt}\right)^2$, where μ is a very small fraction; hence in this case the equation of motion may be written

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = \mu \left(\frac{d\theta}{dt}\right)^2.$$

Since μ is small, as also $\frac{d\theta}{dt}$, we get as a first approximation $\theta = \alpha \cos \sqrt{\frac{g}{l}}\, t$, as before. If this value be substituted in $\mu \left(\frac{d\theta}{dt}\right)^2$, in accordance with the method of successive approximations, the differential equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = \mu\alpha^2 \frac{g}{l} \sin^2 \left(t \sqrt{\frac{g}{l}}\right) = \tfrac{1}{2}\mu \frac{g}{l}\, \alpha^2 \left(1 - \cos 2t \sqrt{\frac{g}{l}}\right),$$

or
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}(\theta - \tfrac{1}{2}\mu\alpha^2) = - \mu \frac{g}{2l}\, \alpha^2 \cos 2t \sqrt{\frac{g}{l}}.$$

The integral of this, subject to the condition that $\theta = \alpha$, and $\frac{d\theta}{dt} = 0$, when $t = 0$, is

$$\theta = \tfrac{1}{2}\mu\alpha^3 + (\alpha - \tfrac{2}{3}\mu\alpha^2) \cos t \sqrt{\frac{g}{l}} + \tfrac{1}{6}\alpha^2\mu \cos 2t \sqrt{\frac{g}{l}}.$$

Also $$\frac{d\theta}{dt} = -\sqrt{\frac{g}{l}} \sin t \sqrt{\frac{g}{l}} \left(\alpha - \tfrac{2}{3}\mu\alpha^2 + \tfrac{1}{3}\mu\alpha^2 \cos t \sqrt{\frac{g}{l}}\right).$$

Hence, since $\frac{d\theta}{dt} = 0$ at the end of one vibration, if T be the corresponding value of t, we have $\sin T\sqrt{\frac{g}{l}} = 0$, or $T = \pi\sqrt{\frac{l}{g}}$. Accordingly the duration of the oscillation is not affected by the resistance. Also, since we have in this case, $\cos t\sqrt{\frac{g}{l}} = -1$, the corresponding value of θ is $-(\alpha - \frac{4}{3}\mu\alpha^2)$; accordingly the resistance of the air reduces the amplitude of the oscillation by $\frac{4}{3}\mu\alpha^2$. The successive angles of oscillation diminish according to the same law, but the time of oscillation remains the same for each.

CHAPTER VI.

WORK AND ENERGY.

116. **Work.**—In all cases where force is employed in overcoming resistance so as to produce motion, work is said to be performed. Hence the conception of work involves both motion and resistance; and therefore a corresponding effort or force to overcome the resistance. In general, work may be defined as the act of producing a change in the configuration of a system in opposition to forces which resist that change. We proceed to consider how the amount of work performed in any case is to be estimated.

117. **Measure of Work.**—The simplest idea of work is derived from raising a weight through a vertical height; in which case the attracting force of the Earth is the resistance overcome. The amount of work in such cases evidently increases in proportion to the weight of the body raised and to the height to which it is raised. For example, the work done in raising one ton through a height of 10 feet is ten times that of raising it one foot, or twenty times that of raising one cwt. through 10 feet; and so on in all cases. Hence it is readily seen that the work performed in such cases is measured by the *product of the weight into the height*, i.e. by Wh, where W represents the number of units in the weight, and h that in the height.

In general, if we confine our attention to a single point which is moved in direct opposition to a constant resisting force, the work done is estimated by the product of the force and the distance through which the point is moved, *i.e.* by Pp, where P represents the force, which overcomes the equal and opposite resisting force, and p the distance passed over.

118. **Gravitation Unit of Work.**—From the ordinary units adopted in this country we derive the unit of work called a *foot-pound*, i.e. the work performed in raising

one pound through one foot in height. This is the unit usually adopted in practical local application of work, and is called the Gravitation Unit of Work (Art. 64). The corresponding unit in the metric system is called the *kilogrammetre*, or kgm. That is the work of raising a kilogramme through the height of a metre. A kilogrammetre is 7·233 foot pounds. The unit of work in this system varies slightly from place to place with the value of *g*, and this should be remembered if numerical or scientific accuracy were required (Art. 38).

119. **Absolute Unit of Work.**—In the absolute system the unit of resistance is that already adopted (Art. 63) as the unit of force. Thus, if we take a poundal as the unit of force, the corresponding unit of work is that done by a poundal acting through a foot. This is sometimes called the foot-poundal. It is obvious that a foot-pound is *g* times a foot-poundal; accordingly any result in the former system is reducible to the latter at any place, by multiplying by the corresponding value of *g*.

Again, adopting the definition of a dyne given in Art. 63, the *work done by a dyne in working through a centimetre* is called an *erg*; and a foot-poundal is 421,394 ergs.

In such measurements as are required in electrical and magnetic investigations, the absolute unit of work is always adopted, and the *erg* is the unit usually employed.

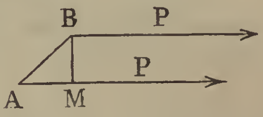
120. **Horse-power.**—Although in our definition of work we have taken no account of the time occupied in its performance, yet time becomes a necessary element when we come to compare the *efficiency* of different agents. For instance, if one agent working uniformly performs an amount of work in one hour which it requires another 5 hours to accomplish, the former is said to be five times as efficient. In comparing the work done by a steam-engine or other agent we usually adopt as our unit the horse-power defined by Watt.

Thus an engine is said to be of one-horse power when it is capable of performing 33,000 foot-pounds of work in one minute of time, or 550 foot-pounds in one second, and so on in proportion.

Continental writers employ horse-power as 75 kgm. per second, which is 542·475 foot-pounds.

121. Again, the work performed in raising a body of weight W to any height h is the same whether the body be raised vertically up or brought up by any other course. The whole work is still represented by Wh, where h is the space through which the weight has been moved, estimated in the vertical direction, *i.e.* in that in which the resistance of gravity acts. And, generally, the work done by any uniform effort or force, acting in a constant direction against an equal and opposite force P, is measured by the product of the force into the space through which its point of application is moved, estimated in the direction in which the force acts.

Thus, if a force P be supposed to act at A, and to move its point of application to B; then if BM be drawn perpendicular to AP, the work done is estimated by Pp, or by $P\Delta s \,.\, \cos\theta$, where $p = AM$, $\Delta s = AB$, and $\theta = \angle BAM$.



The work done is, accordingly, regarded as positive or negative according as the angle θ, which the direction of the force makes with that of the motion, is acute or obtuse.

If $\theta = \frac{1}{2}\pi$, the direction of the motion is perpendicular to that of the force, and the work done is zero.

If two or more forces act on a system, the whole work done is the sum of the works done by each force separately.

If any number of forces be in equilibrium, the total work done by them for any displacement is zero: from this the statical principle of virtual velocities can be readily deduced.

EXAMPLES.

1. Prove that the whole work done in raising a system of heavy bodies, each through a different height, is the same as that of raising the sum of the weights through a height equal to that through which their centre of gravity is raised.

2. Find the work performed in moving a ton along 100 yards on a uniformly rough horizontal road, the coefficient of friction being $\frac{1}{10}$.

Ans. 67,200 foot-pounds.

3. Show that the same work is expended in drawing a body up an inclined plane, subject to friction, as would be expended upon drawing it first along the base of the plane (supposing the coefficient of friction the same), and then raising it up the height of the plane.

4. What time will 10 men take to pump the hold of a ship which contains 30,000 cubic feet of water; the centre of gravity of the water being 14 feet below the point of discharge, and each man being supposed to perform 1500 foot-pounds per minute, assuming the weight of a cubic foot of water to be $62\frac{1}{2}$ lbs.? *Ans.* 29 hrs. 10 mins.

122. **Work done by a Variable Force.**—If the force be not constant, we suppose the path described by its point of application divided into infinitely small portions, through each of which the force may be considered constant. Hence, if the force P act constantly in the direction of the displacement ds of its point of application, Pds is the corresponding *element of work*, and the total work in moving through any space s is represented by the definite integral $\int_0^s Pds$.

If the direction of P makes an angle θ with ds, the corresponding element of work is $P\cos\theta\, ds$, and the total work,

$$\int_0^s P\cos\theta\, ds.$$

Again, let x, y, z, be at any instant the coordinates of the point of application of the force P, referred to a system of rectangular axes; and let X, Y, Z, be the components of P parallel to the coordinate axes respectively; then we have

$$P\cos\theta\, ds = Xdx + Ydy + Zdz.$$

Hence the total work done by P in moving its point of application from one point to another is represented by

$$\int (Xdx + Ydy + Zdz)$$

taken between the two points.

If the expression $Xdx + Ydy + Zdz$ be an exact differential, *i. e.* if

$$Xdx + Ydy + Zdz = du,$$

where u is a function of x, y, z, then the integral

$$\int (Xdx + Ydy + Zdz),$$

taken between any two points, is a function of the coordinates of those points; and the work done is accordingly a function of the extreme coordinates solely. When this is so, the mutual forces between the parts of a system always perform or always consume the same amount of work during any motion whatever by which it can pass from the one particular configuration to the other; hence such a system is called a *conservative system* of forces. In general, for any system of forces acting at different points, the total work done for any finite displacements is represented by

$$\Sigma\int P dp, \text{ or } \Sigma\int(X dx + Y dy + Z dz),$$

where the summation extends to all the forces of the system.

123. **Forces directed to Fixed Centres.**—If we suppose the force F to act from a fixed centre, and if r be the distance of its point of application from the centre, then the corresponding element of work is represented by $F dr$; and the total work, when the point is moved from a distance r' to a distance r'' is represented by $\int_{r'}^{r''} F dr$.

If F be a function of r represented by $\phi'(r)$, then the value of this integral will be

$$\phi(r'') - \phi(r').$$

In the law of attraction which holds in nature we have $F = -\frac{\mu}{r^2}$; and the expression $\mu\left(\frac{1}{r''} - \frac{1}{r'}\right)$ represents the corresponding work, in moving from the distance r' to the distance r''.

Hence the work done in motion from an infinite distance to the distance r is represented by $\frac{\mu}{r}$. The function $\Sigma\frac{m}{r}$ in the case of the ordinary law of gravitation is called the *potential* of the system of attracting masses at the point from which the distances are measured.

Again, if a number of forces F, F', F'', &c., be directed to fixed centres, and if r, r', r'', &c., be the corresponding distances, then the total work is represented by

$$\int F dr + \int F' dr' + \int F'' dr'' + \&c.,$$

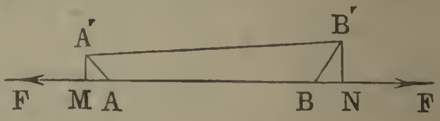
taken between the limiting positions.

If the forces be each a known function of the distance from the corresponding centre of force, the result can, in general, be immediately integrated, and the work is a function of the initial and final positions of the points of application solely. Consequently such a system of forces is always a conservative system.

EXAMPLE.

If m, m' be the masses of two particles attracting each other with a force $\mu\dfrac{mm'}{r^2}$, where r is their distance apart, show that the work done when they have moved from an infinite distance apart to the distance r is $\mu\dfrac{mm'}{r}$.

124. **Work done by a Stress.**—If two equal and opposite forces, each represented by F, act respectively at the points A and B, to find the element of work for a small displacement. Suppose A' and B' to be the new positions for an indefinitely small displacement, and let fall the perpendiculars $A'M$ and $B'N$ on the line AB; then the elements of work are represented by $F \,.\, AM$ and $F \,.\, BN$. Hence their sum is $F(AM + BN) = F(A'B' - AB)$, or $F\Delta s$, where Δs denotes the indefinitely small change in the distance between the points of application of the forces.



Hence, if the points A and B be rigidly connected, as the distance AB is invariable, the total work done by the forces for any displacement is zero.

Also the point of application of a force may be transferred from any one point to any other on its line of action without altering the work done, provided the distance between the two points is invariable.

The pair of equal and opposite forces that two bodies exert on one another in accordance with the general principle of action and reaction is called in modern treatises a *stress*. When the forces act away from each other, as in the figure, the stress is called a *tension*; when they act towards each other it is called a *pressure*.

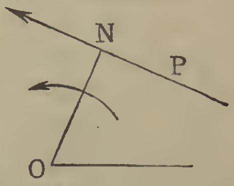
Hence the work done by a stress is positive or negative according as the change of distance between the points of application is in the direction of the mutual action of the forces or in the opposite direction.

Also in the case of a *rigid body* it follows that the *total work done by the internal forces of stress is always zero.*

125. **Body with a Fixed Axis.**—To find the work done by a force acting on a body which is capable of turning round a fixed axis.

Suppose the force resolved into two components, one parallel, the other perpendicular to the fixed axis. The former does no work, since it is perpendicular to the direction of motion of every point in the body.

Let the latter component be represented by P, and suppose it to act in the plane of the paper; the fixed axis being perpendicular to that plane, and meeting it in the point O. Let N be the foot of the perpendicular drawn from O to the line of action of P, then by the last Article we may take N as the point of application of P.



Suppose now the body to receive a small angular displacement $\Delta\theta$ round the fixed axis in the direction of the arrow; then, if $ON = p$, the displacement of N will be $p\Delta\theta$, and the corresponding element of work is $Pp\,\Delta\theta$, or $\Delta\theta$ multiplied by the moment of P with respect to the fixed axis.

Again, if we suppose a pair of equal, parallel, and opposite forces to act on the rigid body; then, provided the plane of the pair is perpendicular to the fixed axis, the work done by the pair is evidently, from what precedes, represented by the moment of the pair multiplied by the small angle of rotation. And if the pair continue to act on the body, the work done by it during any rotation is represented by the product of the moment of the pair by the angle in circular measure, through which the body has rotated.

Example.

A pivot or screw turns round a central axis and presses against a rough plane; find an expression for the work expended on the friction which acts on the circular end of the pivot in one revolution round its axis.

Let Q denote the entire normal pressure between the pivot and the plane, μ the coefficient of friction, supposed constant, a the radius of the end of the pivot. This end may be regarded as consisting of an indefinitely great number of concentric circular rings. If r be the radius of one of the rings, dr its breadth, then the area of the ring is $2\pi r\,dr$, and the corresponding friction, taken over the entire ring, is represented by $\frac{2\mu Q}{a^2}r\,dr$. Hence the corresponding work for one revolution is $\frac{4\mu\pi Q}{a^2}r^2\,dr$. Integrating, we get $\frac{4}{3}\pi\mu Qa$ for the entire work expended. In this investigation the normal pressure Q has been supposed to be uniformly distributed over the end of the pivot.

126. **Energy.**—Energy is the capacity of doing work. For instance, a spring when bent by pressure contains a certain amount of energy stored up in it; thus the mainspring of a watch, by the energy which it possesses, maintains the motions of the works until that energy has been expended. Again, a quantity of air, when compressed into a smaller volume, possesses energy, and can perform work when occasion requires; for example, in projecting a bullet from an air-gun. Also a raised weight is said to possess energy and is capable of doing work. For instance, the motion of a clock is maintained by the energy of its descending weights. This energy of a weight W raised to a height h above the ground is measured by Wh, that is, by the work it is capable of performing by its descent to the ground. In general, when the configuration of a system is altered, it has a tendency to return to its former state, and in effecting this return is capable of doing a certain amount of work. This capacity of doing work, arising from change of configuration or of relative position in a system, is called *potential energy;* the work employed in producing this change being in a sense accumulated. For example, if two bodies which attract one another are separated, they have a tendency to rush together, and in so doing are capable of overcoming a certain amount of resistance.

Again, a body in motion possesses a certain amount of energy which is measured by the work it is capable of performing before being brought to rest. This latter is called the *Kinetic energy* of the body. We proceed to consider how its amount is measured.

127. **Measure of Kinetic Energy.**—The measure of the kinetic energy of the mass m moving, without rotation,

with the velocity v, is easily found. For, suppose the mass acted on by a uniform resistance R in the direction of its motion, and let $R = mf$; then, if v be the initial velocity, and s the space described before coming to rest, we have, by Art. 36, $v^2 = 2fs$; hence $\frac{1}{2}mv^2 = Rs$.

Accordingly, the work which a mass m moving with the velocity v is capable of performing before being brought to rest is $\frac{1}{2}m\,v^2$. Hence its *kinetic energy is equal to half its vis viva*, and is represented by $\frac{1}{2}mv^2$.

Examples.

1. A train of 60 tons, moving at the rate of 15 miles an hour on a horizontal railway, runs, when the steam is shut off and the breaks applied, through a quarter of a mile before stopping. Find in lbs. the mean resistance, and its time of action. *Ans.* 770 lbs.; 2 minutes.

2. The breadth of a river at a certain place is 100 yards, its mean depth is 8 feet, and its mean velocity 3 miles an hour. Calculate its horse power, assuming a cubic foot of water to weigh $62\frac{1}{2}$ lbs.

Here the quantity of water which passes per minute is 633,600 cubic feet; and the required answer is easily seen to be 363 horse-power.

3. A shot of 1000 lbs., moving at 1600 feet per second, strikes a fixed target. How far will the shot penetrate, the target exerting on it an average pressure equal to the weight of 12,000 tons? *Ans.* $1\frac{1}{2}$ ft., approximately.

4. Determine in ergs the kinetic energy of a mass of one hundred pounds moving with a velocity of one foot per minute. *Ans.* 5853.

5. A heavy particle resting on a rough inclined plane, and attached by a string to a fixed point on the plane, is projected from the lowest point of the circle in which it moves in the direction of the tangent; (*a*) find the velocity necessary to carry the string to a horizontal position; (*b*) If the particle descending from this position reach the lowest point and remain there, determine the coefficient of friction.

6. A ball moving with a velocity of 1000 feet per second has its velocity reduced by 100 feet in passing through a plank. Through how many such planks would it pass before being stopped; assuming the same amount of work to be performed in overcoming the resistance of each plank. *Ans.* $5\frac{5}{19}$.

128. **Energy due to a Variable Force.**—If a variable force F act at the centre of gravity of a mass m, in the direction of its motion, we have, by Art. 67,

$$F = m\frac{dv}{dt} = mv\frac{dv}{ds},$$

or

$$Fds = mv\,dv;$$

accordingly, if V_0 and V_1 be the initial and final velocities of m, we have

$$\tfrac{1}{2}m(V_1^2 - V_0^2) = \int_0^s F ds. \tag{1}$$

From this we infer that if a variable force F act on a mass m, in the direction of its motion, the work done by it is measured by half the corresponding change in the *vis viva* of the moving body, or by the change in its kinetic energy.

In general, let X, Y, Z, as before, denote the components parallel to the axes of x, y, z, acting on the mass m, then, by Art. 67, we have

$$X = m\frac{d^2x}{dt^2}, \quad Y = m\frac{d^2y}{dt^2}, \quad Z = m\frac{d^2z}{dt^2}.$$

Multiply the first by dx, the second by dy, and the third by dz, and add; then

$$\begin{aligned} Xdx + Ydy + Zdz &= m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) \\ &= \tfrac{1}{2}md\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right) \\ &= \tfrac{1}{2}md(v^2). \end{aligned}$$

Hence, if V_0 and V_1 be the initial and final velocities,

$$\tfrac{1}{2}m(V_1^2 - V_0^2) = \int (Xdx + Ydy + Zdz),$$

the integral being taken from the initial to the final position of the point. Hence we infer that in this case also the work done by the forces during any motion is equal to half the change in the *kinetic energy* of the moving mass.

If after the lapse of any time the velocity of m become equal to its original value, the work done in that interval by the forces which accelerate the motion is equal to that done by the forces which retard it.

In the case of a central force, represented, as in Art. 123, by $\phi'(r)$, we readily obtain the equation

$$\tfrac{1}{2}m(v^2 - v'^2) = \phi(r) - \phi(r'), \tag{2}$$

where v' denotes the velocity at the distance r' from the centre of force.

Again, in any conservative system of forces the change of kinetic energy in the motion, under the action of the forces, from any one point to any other, is a function of the coordinates of the points, and is independent of the path described.

129. **Equation of Energy**.—In general, if we suppose any free rigid body acted on by external forces, then the total work done by the external forces during any time is equal to the corresponding change of the kinetic energy of the body.

For each particle of the body moves in the same manner as if it were free, and acted on by forces equal to those which result from its connexion with the other particles. Hence, by what precedes, the change in the kinetic energy of the particle is equal to the work done on it by the external forces, together with the work due to the action of the other particles of the body on it. Accordingly, the total change in the kinetic energy of the rigid body in any time is measured by the work done by the external forces in that time, since, by Art. 124, the internal forces in this case do no work, and equation (2) may be written in the generalized form

$$\tfrac{1}{2}\Sigma m(v^2 - v_0^2) = \Sigma\int(Xdx + Ydy + Zdz), \qquad (3)$$

taken between proper limits, in which the sign of summation, Σ, is extended to each element in the body.

Examples.

1. A locomotive of 10 tons, setting out from rest, acquires a velocity of 20 miles an hour on a horizontal railway, after running through a mile under the action of a constant pressure. Calculate in pounds the difference between the moving and retarding forces, approximately. *Ans.* 57.

2. A 50 lb. ball, after traversing the barrel of a gun of 5 feet length, leaves it with a velocity of 500 feet per second. Find approximately the difference between the mean explosive force of the powder and mean resistance which acts on it. *Ans.* 39062·5 lbs.

3. A uniform block of given dimensions stands, with one face perpendicular to the direction of motion, on a railway truck, which is suddenly stopped. If the block be prevented sliding upon the truck, determine the speed of the train so that the block shall be just overturned.

Here the kinetic energy of the block is expended in raising its centre of gravity until it is vertically over the edge round which the block turns. Accordingly, if a be the height of the block, and b the length of its edge which lies in the direction of motion, the required velocity v is given by the equation

$$v^2 = g(\sqrt{a^2 + b^2} - a).$$

4. A catapult is formed by fixing the ends of an elastic string (natural length $2l$) at points A and A', at a short distance apart on a horizontal plane. A bullet placed at the middle point of the string, is drawn back at right angles to AA' (stretched length $= 2l'$), and let go when the string is on the point of breaking. Prove that the velocity V of the bullet when it leaves the string is independent of the distance AA', and is to the velocity V' it would have acquired in falling through the vertical space $l' - l$ in the sub-duplicate ratio of the greatest strain W' the string can bear to the weight W of the bullet. Mr. Whitworth, *Educ. Times.*

Let v be the velocity of the bullet, and $2r$ the length of the string at any instant during the motion; then adopting Hooke's law, that the tension of the string varies directly as its extension, the equation of work becomes

$$\frac{W}{g} V^2 = 4\int_l^{l'} \frac{W'}{l' - l}(r - l)\,dr = 2W'(l' - l),$$

or

$$WV^2 = 2g(l' - l)W' = W'V'^2.$$

5. The following extension of the last question is given by Mr. Townsend. If in place of a single cord there be n uniform cords, of the common unextended length $2l$, attached to as many pairs of diametrically opposite points on the circumference of a fixed circle, and all drawing the bullet along the axis of the cone of which the circle is the base, and the bullet at the vertex, then we shall have

$$WV^2 = 2ngW'(l' - l) = W'V'^2,$$

where V' is the velocity due to the height $n(l' - l)$.

130. **Energy of Rotation.**—To find the kinetic energy of a rigid body revolving round a fixed axis with an angular velocity $\frac{d\theta}{dt}$ or ω.

Let r be the distance from the fixed axis of any element dm of the body; then $r\omega$ will be the velocity of dm, and accordingly the entire *vis viva* of the body,

$$\Sigma v^2 dm = \omega^2 \Sigma r^2 dm = \omega^2 I, \qquad (4)$$

where I represents the moment of inertia of the rigid body relative to the fixed axis (*Int. Calc.*, Art. 196). Thus the kinetic energy required is $\frac{1}{2}I\omega^2$.

EXAMPLES.

1. The rim of a fly-wheel, sp. gr. 7·25, performing 6 revolutions per minute, is 6 inches thick, and its inner and outer radii are 4 and 5 feet respectively: calculate its kinetic energy in foot pounds.

Here $\omega = \frac{\pi}{5}$, and M, the mass of the fly-wheel $= 7{\cdot}25 \times \frac{9}{2} \times 62\frac{1}{2} \,.\, \pi$. lbs. Also (*Int. Calc.*, Art. 201), $I = M(\frac{41}{2})$; hence the required answer is 805 foot pounds, approximately.

2. A rod of uniform density can turn freely round one end; it is let fall from a horizontal position; find its angular velocity when it is passing through the vertical position.

Ans. $\sqrt{\frac{3g}{a}}$, where a is the length of the rod.

3. Two masses M and M' are connected as in Atwood's machine (Art. 71); find the acceleration when the mass μ of the revolving pulley is taken into account. If v be the common velocity of M and M' at any instant, and μk^2 the moment of inertia of the pulley, then the entire *vis viva* of the system is represented by $(M + M')v^2 + \mu k^2 \omega^2$.

Hence, if z be the distance fallen through from rest, we have

$$(M + M')\,v^2 + \mu k^2 \omega^2 = 2g\,(M - M')\,z.$$

Also $v = a\omega$;

$$\therefore\ v^2\{(M + M')\,a^2 + \mu k^2\} = 2ga^2(M - M')z.$$

Again, the acceleration $$f = \frac{dv}{dt} = v\frac{dv}{dz};$$

therefore $$f = \frac{ga^2(M - M')}{(M + M')\,a^2 + \mu k^2}.$$

If the pulley be supposed a homogeneous cylinder, $k^2 = \frac{a^2}{2}$, and f becomes

$$\frac{g(M - M')}{M + M' + \frac{1}{2}\mu}.$$

4. Find in the same case the tensions of the strings.

Ans. $$Mg\,\frac{2M'a^2 + \mu k^2}{(M + M')\,a^2 + \mu k^2},\quad M'g\,\frac{2Ma^2 + \mu k^2}{(M + M')\,a^2 + \mu k^2}.$$

For a homogeneous pulley these become

$$Mg\,\frac{4M' + \mu}{2\,(M + M') + \mu},\quad \text{and}\quad M'g\,\frac{4M + \mu}{2(M + M') + \mu}.$$

5. A homogeneous cylinder, of weight W, is rotating round its axis, supposed horizontal, with an angular velocity ω; find to what height it is capable of raising a given weight P, before coming to rest.

Ans. $\frac{r^2\omega^2}{4g}\,\frac{W}{P}$, where r is the radius of the cylinder.

131. **Vis Viva of any System.**—If $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of gravity of any moving system of masses at any instant, x, y, z the coordinates of the element dm at the same instant; also, if ξ, η, ζ be the coordinates of dm relative to a system of parallel axes drawn through the centre of gravity; then, as in Art. 14, we have, adopting Newton's notation,

$$\dot{x} = \dot{\bar{x}} + \dot{\xi},\quad y = \dot{\bar{y}} + \dot{\eta},\quad \dot{z} = \dot{\bar{z}} + \dot{\zeta};$$

consequently

$$\Sigma v^2 dm = \Sigma dm\{(\dot{\bar{x}} + \dot{\xi})^2 + (\dot{\bar{y}} + \dot{\eta})^2 + (\dot{\bar{z}} + \dot{\zeta})^2\}.$$

Again, if V be the velocity of the centre of gravity, and v' the velocity of dm relative to the centre of gravity, we have

$$V^2 = (\dot{\bar{x}})^2 + (\dot{\bar{y}})^2 + (\dot{\bar{z}})^2, \quad v'^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2,$$

also

$$\Sigma\dot{\xi}\, dm = 0, \quad \Sigma\dot{\eta}\, dm = 0, \quad \Sigma\dot{\zeta}\, dm = 0.$$

Hence we get

$$\Sigma v^2 dm = V^2\, \Sigma dm + \Sigma v'^2 dm. \tag{5}$$

Accordingly, the *vis viva* of the system at any instant consists of two parts, one of which is the *vis viva* of the entire mass supposed concentrated at the centre of gravity; the other is the *vis viva* of the system relative to the centre of gravity regarded as a fixed point.

EXAMPLES.

1. A homogeneous cylinder rolls, without slipping, down a rough inclined plane, under the action of gravity; investigate the motion.

Since the motion is one of pure rolling, the line of contact of the cylinder and plane at any instant may be regarded as fixed; accordingly the friction, acting along the plane, does no work. Also, by Art. 130, the kinetic energy at any instant is represented by $\frac{1}{2}\omega^2 I$, where I is the moment of inertia of the cylinder with respect to the edge in contact with the plane. But $I = M(a^2 + k^2)$, where a is the radius of the cylinder, and Mk^2 its moment of inertia relative to the axis through its centre. Hence the equation of work gives

$$M\omega^2(a^2 + k^2) = 2gMs \sin i,$$

where s is the space down the plane described from rest. Consequently,

$$\omega^2 = \frac{2gs \sin i}{a^2 + k^2}. \quad \text{Also, } \frac{ds}{dt} = v = a\omega;$$

hence

$$\left(\frac{ds}{dt}\right)^2 = \frac{2ga^2 s \sin i}{a^2 + k^2};$$

therefore, by differentiation,

$$\frac{d^2s}{d^2t} = \frac{a^2 g \sin i}{a^2 + k^2}.$$

This determines the acceleration down the plane to be constant. Hence the velocity acquired and the space described in any time can at once be determined. If the cylinder be homogeneous, we have $k^2 = \frac{1}{2}a^2$ (*Int. Calc.*, Art. 201), and the acceleration f in this case is $\frac{2}{3}g \sin i$. This shows that the velocity of the centre of gravity of the cylinder is $\frac{2}{3}$ that acquired by a particle, in the same time, in sliding down a smooth inclined plane of the same inclination. If the cylinder be hollow $k = a$, and accordingly $f = \frac{1}{2}g \sin i$.

2. A mass M draws up another, M', on the wheel and axle; find the motion.

Let a be the radius of the wheel, a' that of the axle; then, as in Ex. 2, Art. 130, it is easily seen that we get

$$\left(\frac{d\theta}{dt}\right)^2 (Ma^2 + M'a'^2 + \mu k^2) = 2g\,(Ma - M'a')\theta + \text{const.}$$

Hence, by differentiation,

$$\frac{d^2\theta}{dt^2} = \frac{g\,(Ma - M'a')}{Ma^2 + M'a'^2 + \mu k^2}.$$

Accordingly, if $\theta = 0$, and $\dfrac{d\theta}{dt} = 0$, when $t = 0$, we get for the angle turned through in the time t,

$$\theta = \tfrac{1}{2}gt^2 \frac{Ma - M'a'}{Ma^2 + M'a'^2 + \mu k^2}.$$

3. Find the tensions of the strings in the same case.

Ans. $Mg\dfrac{M'a'(a + a') + \mu k^2}{Ma^2 + M'a'^2 + \mu k^2}$, $M'g\dfrac{Ma(a + a') + \mu k^2}{Ma^2 + M'a'^2 + \mu k^2}$.

4. Find the velocity acquired by the centre of a hoop in rolling down an inclined plane of height h. *Ans.* $\sqrt{gh}$.

132. **Work done by an Impulse.**—If a mass M moving with a velocity V receives an impulse in the direction of its motion, and if V' be its velocity after the impulse, then the change in its kinetic energy is

$$\tfrac{1}{2}M(V'^2 - V^2) = M(V' - V) \cdot \tfrac{1}{2}(V' + V).$$

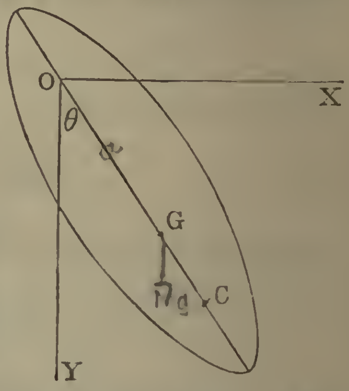
But $M(V' - V)$ measures the impulse. Hence the work done by the impulse is measured by the product of the momentum, which measures the impulse, by half the sum of the velocities before and after the impulse.

For example, a bullet m in passing through a plank experiences a definite amount of resistance, measured by the thickness and by the resisting force, but this equals half the loss of *vis viva* of the bullet, or

$$\tfrac{1}{2}m(v^2 - v'^2) = m(v - v') \cdot \tfrac{1}{2}(v + v'),$$

where v and v' are the velocities with which it meets and leaves the plank. Hence the momentum $m(v - v')$ communicated to the plank varies inversely as $v + v'$: consequently the greater the velocity of impact the less the momentum imparted. This explains how a bullet with a high velocity can pass through a door without moving it on its hinges.

133. **Compound Pendulum.**—A solid body oscillating under the action of gravity, around a fixed horizontal axis, is called a *compound pendulum.* The motion of such a body is readily reduced to that of the corresponding simple pendulum, as follows:



Let the plane of the paper represent that in which the motion of G, the centre of gravity of the body, takes place, and let O be the point in which the fixed axis intersects that plane. Draw OY vertically downwards, and let $GO = a$, M = mass of the body. Also let $\angle GOY = \theta$.

Suppose the pendulum to start from rest, when $\theta = \alpha$, then, in the time t, the point G will have descended through the vertical height $a(\cos\theta - \cos\alpha)$. Also the *vis viva* of the body at the same instant (Art. 130) is represented by

$$\left(\frac{d\theta}{dt}\right)^2 \Sigma r^2 dm = I\left(\frac{d\theta}{dt}\right)^2.$$

Hence, by the principle of work, Art. 129, we have

$$I\left(\frac{d\theta}{dt}\right)^2 = 2Mga(\cos\theta - \cos\alpha).$$

If the moment of inertia I be represented by MK^2, the latter equation becomes

$$K^2\left(\frac{d\theta}{dt}\right)^2 = 2ga(\cos\theta - \cos\alpha), \qquad (6)$$

where K is the radius of gyration of the body (*Int. Calc.*, Art. 197), relative to the axis of suspension.

Hence, by differentiation,

$$\frac{d^2\theta}{dt^2} + \frac{ga}{K^2}\sin\theta = 0. \tag{7}$$

Comparing this with the corresponding equation for the motion of a simple pendulum (Art. 99), we see that the motion is the same as that of a simple pendulum of length $l = \frac{K^2}{a}$.

Again, if Mk^2 be the moment of inertia relative to an axis through the centre of inertia parallel to the axis of suspension, we have (*Int. Calc.*, Art. 196),

$$K^2 = a^2 + k^2;$$

hence

$$l = \frac{K^2}{a} = a + \frac{k^2}{a}. \tag{8}$$

The point O is called the *centre of suspension*. If OG be produced until $OC = l$, since the body moves as if its entire mass were concentrated at the point C, that point is called the *centre of oscillation*. Again, if through C a right line be drawn parallel to the axis of suspension, all the points of this line move like the point C, i.e. as if they were freely suspended from the axis of rotation. This line is called the *axis of oscillation*.

Again, since $OG \, . \, GC = k^2$, the axes of suspension and oscillation are interchangeable, *i.e.* the time T of an oscillation is the same for both, viz., $T = \pi\sqrt{\frac{a^2 + k^2}{ag}}$.

By varying the axis of suspension the time of a small oscillation will also, in general, vary.

For parallel axes, T obviously is a minimum when $a = k$, and the corresponding time of a small oscillation $= \pi\sqrt{\frac{2k}{g}}$.

In order that this should be the smallest possible, the axis of suspension must be parallel to that axis round which the moment of inertia is least (*Int. Calc.*, Art. 217).

If the axis of suspension of a compound pendulum be inclined at an angle α to the vertical, it is readily seen that the preceding investigation holds good, provided $g \sin \alpha$ be substituted for g throughout.

Again, as in Art. 99, the time of any motion of a compound pendulum is represented by an elliptic integral.

Also if a solid body make a complete revolution round a horizontal axis, the time of revolving through any angle can be reduced to that for the corresponding oscillatory motion of a particle.

Examples.

1. A uniform circular plate, of radius a, makes small oscillations about a horizontal tangent; find the length of the equivalent simple pendulum. *Ans.* $\frac{5}{4}a$.

2. Find the position of the axis with respect to which a uniform circular plate will oscillate in the shortest time.
Ans. The axis is at a distance of half the radius from the centre. Length of the equivalent pendulum $= a$.

3. Find the centre of oscillation of a homogeneous sphere, of radius a, oscillating round a horizontal tangent to its surface.
Ans. At a point $\frac{2}{5}a$ below the centre.

4. Find the ratio of the times of oscillation of a homogeneous solid sphere, and of a spherical shell of equal diameter, each being taken with reference to a horizontal tangent. *Ans.* $\sqrt{21} : 5$.

5. A sphere of radius a is suspended by a fine wire from a fixed point, at a distance l from its centre; prove that the time of a small oscillation is represented by $\pi\sqrt{\dfrac{5l^2 + 2a^2}{5lg}}\,(1 + \frac{1}{4}\sin^2\frac{1}{2}\alpha)$, where α represents the amplitude of the vibration.

6. If the semiaxes of a uniform elliptic disc be 2 feet and 1 foot, and it be suspended from an axis perpendicular to its plane through one of its foci, find the time of a complete oscillation under gravity. *Ans.* $\dfrac{\pi}{2\sqrt[4]{3}}\sqrt{\dfrac{17}{g}}$.

134. **Determination of the Force of Gravity.**—We have already seen (Art. 101), that the value of g at any place can be determined from the length of the seconds pendulum at the place. To apply this it is necessary to know the numerical value of $\dfrac{a^2 + k^2}{a}$.

Two methods have been devised for this purpose—one employed by Borda, Arago, Biot, and others; the other first used by Bohnenberger, and afterwards brought to great perfection by Captain Kater.

In the first method the compound pendulum, supposed made of a material of uniform density, has such a shape that its radius of gyration can be calculated mathematically, as also the distance of its centre of inertia from the fixed axis.

The second method depends on the reciprocity of the centres of suspension and oscillation.

Kater's compound pendulum consisted of a heavy bar having two apertures at opposite sides of the centre of inertia, through which knife edges passed, on either of which the body could be supported. On the bar was placed a ring capable of being moved up or down by means of a screw. Kater moved the ring until the times of oscillation round the two axes were equal; in which case, by the preceding, the distance between the axes is equal to the length of the equivalent simple pendulum. The distance, l, between the axes having been accurately measured, the value of g was calculated from the formula $g = \frac{\pi^2 l}{T^2}$, where T denotes the time of an oscillation.

Kater published an account of his observations in the *Philosophical Transactions*, 1818, 1819. For a more detailed account of this method the reader is referred to Routh's *Rigid Dynamics*, Arts. 100–108.

135. **Motion of a Rigid Body round a Fixed Axis.**—In general, let a force P, in a direction which is at right angles to the fixed axis act on a body, then for a small angular motion $d\theta$ the work done by P is, by Art. 125, represented by $Pp\,d\theta$. Again, as this work is equal to the corresponding change in the *kinetic* energy of the body, we have

$$Pp\,d\theta = \tfrac{1}{2} Mk^2 d\left(\frac{d\theta}{dt}\right)^2 = Mk^2 \frac{d^2\theta}{dt^2}\,d\theta.$$

Hence we get

$$\frac{d^2\theta}{dt^2} = \frac{Pp}{Mk^2} = \frac{\text{Moment of impressed force}}{\text{Moment of inertia}}. \qquad (9)$$

EXAMPLES.

1. A uniform circular plate of 1 foot radius and 1 cwt. revolves round its axis 5 times per second; calculate its kinetic energy in foot pounds.

Ans. 863, approximately.

2. A bent lever ACB rests in equilibrium when AC is inclined at the angle α to the horizontal line; show that when this arm is raised to the horizontal position it will fall through the angle 2α, C being supposed fixed.

3. A homogeneous cylinder, of mass M, and radius a, turns round a horizontal axis; a fine thread is wrapped round it, and has a mass M' attached to its extremity. Find the angular velocity of the cylinder when M' has descended through the height h.

Ans. $\omega^2 = \dfrac{4M'gh}{a^2(M + 2M')}$.

4. A right cone oscillates round a horizontal axis, passing through its vertex and perpendicular to the axis of the cone; find the length of the equivalent simple pendulum.

Ans. $\dfrac{4h^2 + b^2}{5h}$, where h is the height of the cone, and b the radius of its base.

5. If in the last example the cone be let fall from the position in which its axis is horizontal, find its angular velocity when in the lowest position.

Ans. $\omega^2 = \dfrac{10hg}{4h^2 + b^2}$.

6. In the same case find the pressure on the fixed axis, at the lowest position of the body, arising from centrifugal force (Art. 95).

Ans. $\dfrac{15}{2} W \dfrac{h^2}{4h^2 + b^2}$, where W represents the weight of the cone.

7. A thin beam, whose mass is M and length $2a$, moves freely about one extremity attached to a fixed point whose distance from a smooth plane is b, $(b < 2a)$: the other extremity rests on the plane, the inclination of which is α. If the beam be slightly displaced from its position of equilibrium, determine the time of its small oscillations.

Ans. $\pi \sqrt{\dfrac{4a^2 - b^2}{3ag \sin \alpha}}$.

Indian Civil Service Exam., 1860.

In this case the beam may be regarded as turning round the perpendicular on the plane.

8. A bullet weighing 50 grammes is fired into the centre of a target with a velocity of 500 metres a second. The target is supposed to weigh a kilogramme, and to be free to move. Find, in kilogrammetres, the loss of energy in the impact.

Lond. Univ., 1880. *Ans.* 635·6.

9. When the weight P of the pulley is taken into account, show that equation (9), Art. 74, becomes

$$\mu = \frac{W}{W'} - \frac{f}{g}\left(1 + \frac{W + \frac{1}{2}P}{W'}\right),$$

in which the pulley is supposed to be of uniform density and thickness.

10. If the motion of a solid body acted on by attracting forces be a pure rotation, the velocity ω of rotation at any instant will be given by the equation

$$Mk^2(\omega^2 - \omega_0^2) = 2(V - V_0),$$

where V represents the potential of the attracting forces.

11. A hollow cylinder rolls down a perfectly rough inclined plane in 10 minutes; find the time a uniform solid cylinder would take to roll down the same plane. *Ans.* $5\sqrt{3}$ minutes.

12. The particles composing a homogeneous sphere of mass M and radius r were originally at an infinite distance from each other. Prove that the work done by their mutual attraction is $\frac{3}{5}\frac{M^2}{r}$.

13. Apply the principle of work to the investigation of the loss of kinetic energy in the collision of imperfectly elastic bodies, in terms of their moduli of elasticity.

When the mass M is in collision with M', by the third law of motion, the pressures developed in the two are always equal and opposite. If P denote this pressure at any instant, the changes in velocity in that instant are $\frac{P}{M}$ and $\frac{P}{M'}$. As this is true for each instant, the total change of velocity in one body during any portion of the collision will be to the total change of velocity in the other inversely as the masses. Hence, as before (Art. 76), we have for the common velocity u, at the instant of greatest compression, $\frac{V-u}{u-V'} = \frac{M'}{M}$. Again, for the velocities v, v' after impact, $MV + M'V' = Mv + M'v'$. Supposing the bodies imperfectly elastic, the impressions made by the impact are only partially done away with afterwards, and so a part of the work expended on them is lost, as regards motion of the bodies. Call this loss of work W, then the equation of energy is

$$MV^2 + M'V'^2 = Mv^2 + M'v'^2 + 2W.$$

Let P be the pressure or the opposite resistance which produces the change of form, and let s, s' be the amounts of the compressions in the two bodies in the direction of impact; then the amount of work employed in the compression, since the resistance P gradually increases from 0 to P, is $\frac{1}{2}P(s + s')$.

Again, if E, E' be the moduli of elasticity, l, l' the lengths, and σ, σ' the cross sections of the impinging bodies; then, by Hooke's law, we may write $s = \frac{Pl}{\sigma E}$, $s' = \frac{Pl'}{\sigma' E'}$; hence the quantity of work employed in the compression of the bodies is $\frac{1}{2}P^2\left(\frac{l}{\sigma E} + \frac{l'}{\sigma' E'}\right)$. At this stage the two have the same velocity, $u = \frac{MV + M'V'}{M + M'}$; thus we have for the loss of kinetic energy up to this

$$\tfrac{1}{2}(MV^2 + M'V'^2) - \tfrac{1}{2}u^2(M + M') = \tfrac{1}{2}(V - V')^2\,\frac{MM'}{M + M'}.$$

Equating this to the work, we can find the greatest pressure P which, produced by the collision, effects the change in form of the two bodies

$$P = (V - V')\sqrt{\frac{MM'}{M + M'} \cdot \frac{\sigma E\,\sigma' E'}{l\sigma' E' + l'\sigma E}}.$$

Now suppose in the second stage of the collision that M recovers the fraction n, and M' the fraction n', of the work employed in change of form, then the work lost is

$$W = \tfrac{1}{2}P\{(1-n)s + (1-n')s'\} = \tfrac{1}{2}P^2\left\{\frac{(1-n)l}{\sigma E} + \frac{(1-n')l'}{\sigma' E'}\right\}.$$

Putting in its value for P we get

$$W = \tfrac{1}{2}(V - V')^2\,\frac{MM'}{M + M'}\,\frac{(1-n)l\sigma' E' + (1-n')l'\sigma E}{l\sigma' E' + l'\sigma E}.$$

Comparing with Ex. 7, p. 77, we may therefore write

$$e^2 = \frac{nl\sigma' E' + n'l'\sigma E}{l\sigma' E' + l'\sigma E}.$$

Thus for two bodies of equal elasticity $n = n' = e^2$.

CHAPTER VII.

CENTRAL FORCES.

SECTION I.—*Rectilinear Motion.*

136. **Centre of Force.**—We next proceed to consider motion under the action of a force whose direction always passes through a fixed point, and whose intensity is a function of the distance from that point. The fixed point is called the *Centre of Force ;* and the force is said to be attractive or repulsive according as it is directed towards or from the centre.

If we assume that two particles of equal mass placed at the same distance from a centre of attractive force are equally attracted towards the centre, when they are conceived placed *together*, the whole force acting on them—considered as one mass—will be double that which acts on one of the particles. Similarly, if any number (n) of equal particles be placed together, the whole force will be n times that which acts on a single particle. Hence it follows that in such cases the whole attracting force is proportional to the number of particles, *i.e.* to the *mass* of the attracted body—provided the attracted mass be of such small dimensions that the lines drawn from its several points to the centre of force may be regarded as equal and parallel. Accordingly the force, in this case, is proportional to the attracted mass, and consequently the *acceleration* produced by it is independent of the mass attracted, and is a function of the distance from the centre of force only.

137. **Attraction.**—The acceleration due to an attractive force, at any distance, is called the *attraction* of the force, and is, as we have seen, independent of the mass of the attracted particle. Consequently the measure of an attractive force at any distance is the velocity per second which the

central force could generate in one second, in its own direction, if it were conceived to act uniformly during that time. For instance, g, i.e. the velocity acquired in one second by a falling body (Art. 36), measures the attractive force of the Earth, at any place, and is, as already stated, the same for all bodies at that place.

138. **Rectilinear Motion.**—If the particle acted on be originally at rest, or be projected in the line joining its position to the centre of force, its motion will take place in that right line.

Taking this line for the axis of x, and the fixed centre as origin, we have for the equation of motion (Art. 21),

$$\frac{d^2x}{dt^2} = -F, \qquad (1)$$

where F represents the *attraction* at the distance x, which is taken with the negative sign because it tends to *diminish* the velocity.

We shall illustrate equation (1) by applying it to a few elementary cases.

139. **Force Varying as the Distance.**—If the force be proportional to the distance from the fixed centre, we may assume $F = \mu x$; then, for attractive forces, the equation of motion becomes

$$\frac{d^2x}{dt^2} = -\mu x, \text{ or } \frac{d^2x}{dt^2} + \mu x = 0. \qquad (2)$$

This equation has been already considered in Art. 107, and accordingly we have

$$x = C \cos t\sqrt{\mu} + C' \sin t\sqrt{\mu}. \qquad (3)$$

The constants C and C' are determined from the initial circumstances of the motion.

For example, if the particle start from rest, at the distance a from the centre of force; then, when $t = 0$, we have $x = a$, and $\frac{dx}{dt} = 0$: this gives

$$C = a, \text{ and } C' = 0;$$

and consequently $x = a \cos t\sqrt{\mu}$. This determines the position of the particle at any instant.

Again, if $(t' - t)\sqrt{\mu} = 2\pi$, it is evident that the values of x and of $\frac{dx}{dt}$ are the same at the end of the time t' as at the time t: this shows that the motion is oscillatory, and that the time of a complete vibration is $\frac{2\pi}{\sqrt{\mu}}$. (Compare Art. 109.)

For a repulsive force the equation of motion is

$$\frac{d^2x}{dt^2} = \mu x. \tag{4}$$

Accordingly (Art. 107), we have

$$x = Ce^{t\sqrt{\mu}} + C'e^{-t\sqrt{\mu}}.$$

To determine the constants: suppose, as in the former case, the particle starts from rest, at the distance a; then

$$a = C + C', \text{ and } C - C' = 0.$$

Hence

$$x = \tfrac{1}{2}a\left(e^{t\sqrt{\mu}} + e^{-t\sqrt{\mu}}\right). \tag{5}$$

140. **Inverse Square of Distance.**—In the case of the law of nature, in which the attractive force varies as the inverse square of the distance, we have $F = \frac{\mu}{x^2}$; and the differential equation of motion is

$$\frac{d^2x}{dt^2} + \frac{\mu}{x^2} = 0.$$

Multiplying by $2dx$, and integrating, we get

$$\left(\frac{dx}{dt}\right)^2 - \frac{2\mu}{x} = \text{const.}$$

Hence, if the particle be supposed to start from rest, at the distance a,

$$\left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right). \qquad (6)$$

This equation determines the velocity at any distance from the centre of force.

Again, extracting the square root, and transforming, we get

$$\sqrt{2\mu}\,dt = \frac{-\,dx}{\sqrt{\frac{1}{x} - \frac{1}{a}}}. \qquad (7)$$

The negative sign is taken since in the motion towards the centre of force, x diminishes as t increases.

To integrate this equation, assume $x = a\cos^2\theta$; then

$$\sqrt{\frac{1}{x} - \frac{1}{a}} = \frac{\tan\theta}{\sqrt{a}}, \text{ and } dx = -\,2a\sin\theta\cos\theta\,d\theta;$$

consequently $\qquad \sqrt{2\mu}dt = 2a^{\frac{3}{2}}\cos^2\theta\,d\theta;$

hence $\qquad t = \sqrt{\frac{a^3}{2\mu}}\,(\theta + \tfrac{1}{2}\sin 2\theta) + \text{constant}.$

Again, the constant vanishes, since t and θ vanish, when $x = a$;

$$\therefore\ t = \sqrt{\frac{a^3}{2\mu}}\,(\theta + \tfrac{1}{2}\sin 2\theta). \qquad (8)$$

Hence the time of motion from the distance a to the distance x is

$$t = \sqrt{\frac{a}{2\mu}}\left(a\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{x(a-x)}\right). \qquad (9)$$

Also the time of motion to the centre of force is

$$\frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}}.$$

Again, if the body be supposed to start from an indefinitely great distance we have, making $a = \infty$ in (6),

$$v^2 = \frac{2\mu}{x}. \qquad (10)$$

141. **Application to the Earth.**—For the law of nature, it is easily proved that the attraction of a homogeneous sphere is the same as if its mass were concentrated at its centre (Minchin's *Statics*, Art. 244). This property holds also for *heterogeneous spheres*, provided the density at each point is a function of its distance from the centre solely; for the spheres can be supposed divided into an indefinite number of homogeneous spherical shells, each of which attracts as if its mass were concentrated at its centre. Hence, the results of the last Article can be readily applied to the motion of a body falling from any height above the Earth's surface, all resistance of the atmosphere being neglected.

In this case g measures the Earth's attraction at its surface; hence, if R denote the Earth's radius, we have $\mu = gR^2$, and if this value be substituted for μ we can readily determine the velocity and time of motion in any particular case.

For instance, the velocity V with which a body falling from the height h would reach the surface of the Earth is given by the equation

$$V^2 = 2g\frac{Rh}{R+h}. \qquad (11)$$

Also, by (9), the time of motion in seconds is

$$\sqrt{\frac{R+h}{2g}}\left\{\frac{R+h}{R}\sin^{-1}\sqrt{\frac{h}{R+h}} + \sqrt{\frac{h}{R}}\right\},$$

where R and h are expressed in feet.

If $R = nh$, this becomes

$$\sqrt{\frac{h}{2g}\frac{(1+n)}{n}}\left\{\frac{1+n}{\sqrt{n}}\sin^{-1}\frac{1}{\sqrt{1+n}} + 1\right\}.$$

When n is a large number this becomes, approximately,

$$\sqrt{\frac{2h}{g}}\left(1+\frac{5}{6n}\right). \tag{12}$$

If the body be supposed to start from an infinite distance, the velocity with which it would reach the Earth is given by the equation

$$v^2 = 2gR. \tag{13}$$

142. **Comparison of Attraction of Different Spherical Bodies.**—Let M, M' denote the masses of two spheres; δ, δ' their mean densities; r, r' their radii; f, f' their attractions at their surfaces, respectively: then we have

$$f : f' = \frac{M}{r^2} : \frac{M'}{r'^2} = \delta r : \delta' r'.$$

For example, if D be the mean density of the Earth, and R its radius, then f, the attraction at the surface of a planet of radius r and mean density δ, is given by the equation

$$f = g\frac{\delta}{D}\frac{r}{R}. \tag{14}$$

If the mean densities be the same for both, we have

$$f = g\frac{r}{R}.$$

If we assume the mean density of the Sun to be one-fourth that of the Earth, and its radius 104 times that of the Earth, then the velocity acquired in one second by a falling body at the Sun's surface is approximately represented by $26g$.

In the case of the mutual attraction of two spheres it is often convenient to assume the origin at their common centre of gravity, which remains a fixed point during the motion. For instance, if two equal spheres, each of radius r, be placed at a given distance apart, and left to their mutual attraction, to find the time they would take to come together.

Let $2a$ be the initial distance between their centres, and assume the origin O at the middle point of the line joining the centres. If x be the distance of the centre of either sphere from O at any time; then $\frac{\mu}{4x^2}$ represents the corresponding attraction, and the time required is, by (9), represented by the expression

$$\sqrt{\frac{2a}{\mu}}\left(a\cos^{-1}\sqrt{\frac{r}{a}}+\sqrt{r(a-r)}\right), \qquad (15)$$

where μ can be determined by the equation

$$\frac{\mu}{r^2}=f=g\,\frac{\delta}{D}\,\frac{r}{R}.$$

EXAMPLES.

1. If h be the height due to the velocity V at the Earth's surface, supposing its attraction constant, and H the corresponding height when the variation of gravity is taken into account, prove that

$$\frac{1}{h}-\frac{1}{H}=\frac{1}{R}.$$

2. If a man weigh 10 stone on the Earth's surface, calculate, approximately, his weight if he were transferred to the surface of the Sun.

Ans. 1 ton, 13 cwt.

3. Calculate, approximately, the velocity with which a body falling from an indefinitely great distance would reach the surface of the Earth; neglecting all forces besides the Earth's attraction, and assuming $R = 4000$ miles.

Ans. 7 miles per second.

4. Calculate in like manner the velocity with which a body falling from an indefinitely great distance would reach the surface of the Sun.

Ans. 364 miles.

5. In a work erroneously attributed to Sir Isaac Newton, it is stated, that if two spheres, each one foot in diameter, and of a like nature to the Earth, were distant by but the fourth part of an inch, they would not, even in spaces void of resistance, come together by the force of their mutual attraction in less than a month's time.

Investigate the truth of this statement. *Sch. Ex.*, 1883.

Equation (15) gives in this case for the time, in seconds,

$$700\sqrt{11}\left\{\frac{49}{96}\sin^{-1}\left(\frac{1}{7}\right)+\frac{1}{8\sqrt{3}}\right\}.$$

This gives about 5 minutes and 38 seconds.

If the question be solved on the assumption that the attraction is constant during the motion, and equal to that when the spheres are touching, the time required is readily found to be, approximately $= 100\sqrt{11} = 5$ m. 32 secs.

It may be observed that the former result follows from this immediately by application of formula (12).

6. Show that if a sphere, of the same density as the Earth, attract a particle placed at the n^{th} part of its radius from its surface, the time of motion to the surface is the same as that of a particle moving to the Earth from a distance equal to the n^{th} part of its radius.

7. What is meant by the Astronomical Unit of Mass? The astronomical unit of mass is that mass which attracts a particle placed at the unit of distance so as to produce in it the unit of acceleration in the unit of time.

8. If a foot and a second be taken as the units of length and time, calculate approximately the number of pounds in the astronomical unit of mass.

Let M denote the mass of the Earth, and m that of the astronomical unit, then we have

$$\frac{M}{r^2} : m = g : 1, \quad \text{or} \quad m = \frac{M}{gr^2},$$

where r is the radius of the Earth in feet. Now assuming D, the mean density of the Earth, to be $5\frac{1}{2}$ that of water, the mass of a mean cubic foot of the Earth is 344 lbs. approximately. If we assume the radius of the Earth to be 4000 miles, we get

$$\frac{M}{gr^2} = \frac{4}{3}\frac{\pi r}{g} \times 344 = 951{,}000{,}000 \text{ lbs., approximately.}$$

9. Taking the value of gravity as 981 in centimetres and seconds, and the Earth's radius as $6·37 \times 10^8$ centimetres: find the Earth's mass in astronomical units. *Ans.* 398×10^{18}.

143. **Force any Function of Distance.**—If the force be attractive and vary inversely as the n^{th} power of the distance, the equation of motion becomes

$$\frac{d^2x}{dt^2} + \frac{\mu}{x^n} = 0.$$

Multiply, as before, by $2dx$, and integrate; then

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{n-1}\frac{1}{x^{n-1}} + \text{const.},$$

or

$$v^2 = \frac{2\mu}{n-1}\frac{1}{x^{n-1}} + \text{const.}$$

If the attracted particle start from rest at the distance a, we have

$$v^2 = \frac{2\mu}{n-1}\left(\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right). \tag{16}$$

This determines the velocity at any distance from the centre.

In general, if $F = \mu\phi'(x)$, we have

$$\frac{d^2x}{dt^2} + \mu\phi'(x) = 0;$$

and, proceeding as before, we get

$$\left(\frac{dx}{dt}\right)^2 + 2\mu\int\phi'(x)\,dx = \text{const.}$$

If V denote the velocity at the distance a, this gives

$$v^2 - V^2 = 2\mu\{\phi(a) - \phi(x)\}. \tag{17}$$

If the body start from rest at the distance a, its velocity at any distance x is given by the equation

$$v^2 = 2\mu\{\phi(a) - \phi(x)\}. \tag{18}$$

144. **Elastic Strings.**—We next proceed to consider a few simple cases of rectilinear motion for heavy bodies attached to elastic strings.

We assume in all cases Hooke's law, that *the tension of the string is proportional to its extension beyond its natural length*, to be applicable throughout the motion; and we *neglect the weight of the string.*

Let us commence with the following example:—

One end of an elastic string is attached to a fixed point on a smooth horizontal table, and the other end to a particle, of mass m, on the table. If the string be extended beyond its natural length, and then let go, to find the subsequent motion of the particle.

Let a be the natural length of the string, x its length at any instant during the motion; then $x - a$ represents its extension at that instant.

Again, let b represent the extension when we suppose the string to hang freely supporting the given particle; then, by Hooke's law, the tension T of the string for the extension $x-a$ i srepresented by

$$T = mg\frac{x-a}{b}. \tag{19}$$

Accordingly, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0. \tag{20}$$

Integrating, we have

$$x = a + C\cos\sqrt{\frac{g}{b}}\,t + C'\sin\sqrt{\frac{g}{b}}\,t.$$

To determine the constants, let a' denote the *initial length* of the string; then

$$a' = a + C, \text{ i.e. } C = a' - a;$$

also, since $\frac{dx}{dt} = 0$ when $t = 0$, we have

$$C' = 0.$$

Consequently $x = a + (a' - a)\cos\sqrt{\frac{g}{b}}\,t.$ (21)

This gives the position of the particle so long as the string is stretched, *i.e.* so long as x is greater than a.

The velocity at any instant is given by the equation

$$\frac{dx}{dt} = -(a' - a)\sqrt{\frac{g}{b}}\sin\sqrt{\frac{g}{b}}\,t.$$

The length x becomes equal to a, or the string regains its natural length, and the tension ceases to act at the end of the time $\frac{\pi}{2}\sqrt{\frac{b}{g}}$.

Meanwhile the velocity has increased from zero and attained its maximum value

$$-(a' - a)\sqrt{\frac{g}{b}},$$

at the same instant.

The particle will now continue to move uniformly along the table with this velocity until it arrives at the same distance a on the opposite side of the fixed extremity of the string, when it becomes again acted on by the retarding tension of the string; and the same motion will be repeated.

145. **Weight Suspended by an Elastic String.**—We shall next consider the vertical oscillations of a body, of weight W, attached to the end of an elastic string, which hangs freely from a fixed point. Suppose the body depressed below the position of equilibrium, and then set at liberty, to investigate the subsequent motion.

As before, let b be the extension of the string due to the weight W; c its extension at the commencement of the motion; x its extension at any instant; T the corresponding tension of the string: then, by Hooke's law, we have

$$T = W\frac{x}{b}, \qquad (22)$$

and the differential equation of motion is obviously

$$m\frac{d^2x}{dt^2} = W - T,$$

or

$$\frac{d^2x}{dt^2} + \frac{g}{b}(x - b) = 0. \qquad (23)$$

The integral of this is

$$x = b + C\cos\sqrt{\frac{g}{b}}\,t + C'\sin\sqrt{\frac{g}{b}}\,t.$$

To determine the constants, we have, when

$$t = 0, \ x = c, \ \text{and} \ \frac{dx}{dt} = 0;$$

therefore

$$C = c - b, \ \text{and} \ C' = 0.$$

Consequently

$$x = b + (c - b)\cos\sqrt{\frac{g}{b}}\,t. \qquad (24)$$

There are two cases to be considered, according as c is less or greater than $2b$.

(1). Let $c < 2b$. In this case the extension x, and consequently the tension T, can never vanish; and the body will oscillate up and down through the distance $c - b$, on both sides of the position of equilibrium; the time of an oscillation being represented by

$$\pi\sqrt{\frac{b}{g}}.$$

(2). Next, let $c > 2b$. In this case x vanishes, and consequently T also, when

$$b + (c - b)\cos\sqrt{\frac{g}{b}}\,t = 0.$$

The corresponding velocity is easily found to be

$$\sqrt{\frac{gc(c - 2b)}{b}}.$$

As the tension of the string vanishes at this instant, the body may be regarded as projected upwards with the foregoing velocity. The height, h, to which it would ascend is given by the equation

$$h = \frac{c}{2b}(c - 2b). \qquad (25)$$

The body will afterwards fall to the origin, and the subsequent motion will be as before.

146. **Weight Dropped from a Height.**—Next suppose the weight attached to the string, and dropped from a height h, vertically above the lower extremity of the string when hanging freely and unstretched. The solution is contained in the preceding investigation: for the maximum extension c of the string is given by (25), and is represented by

$$c = b + \sqrt{b(b + 2h)}. \qquad (26)$$

In practice it is found that Hooke's law does not hold beyond certain limits, which are attained long before the string is broken. It is interesting to consider whether in any particular case the string will be broken or not by the fall, assuming Hooke's law still to hold.

A given string is capable of supporting only a certain weight, called its *breaking weight*. Denote this weight by B;

then e, the corresponding extension of the string, is found, by Hooke's law, from

$$B = \frac{W}{b}\,e, \tag{27}$$

and the string will break or not according as the maximum extension, given by the preceding analysis, is greater or less than c; that is, according as $b + \sqrt{b\,(b + 2h)}$ is greater or less than e.

Again, if b and e be both given, the least height of fall, h, in order that the string should break, is got by substituting e for c in (25), and is

$$h = \frac{e\,(e - 2b)}{2b}. \tag{28}$$

Suppose the weight W to be the n^{th} part of B, *i.e.* let $e = nb$, and we have $h = e\,(\frac{1}{2}n - 1)$.

Thus, for instance, a weight $\frac{1}{4}$ of the breaking weight, dropped from the height e, should suffice to break the string.

The preceding analysis applies also to the vertical oscillations of rods supporting heavy weights; and many interesting practical questions are explained thereby—for instance, the danger to the stability of a suspension bridge arising from the steady march of troops over it.—See Poncelet, *Mécanique Industrielle*, Arts. 332–345.

Examples.

1. A heavy particle attached to a fixed point by an elastic string is allowed to fall freely from this point. Show that the elastic force at the lowest point is given by the equation

$$F = 2W\frac{\text{total fall}}{\text{extension of string}},$$

where W is the weight of the particle.

2. A heavy particle attached to a fixed point by an elastic string hangs freely, stretching the string by a quantity e. It is drawn down by an additional distance f; determine the height to which it will rise if $f^2 - e^2 = 4ae$, a being the unstretched length of the string. *Ans.* $2a$.

3. A heavy body is attached to a fixed point by an elastic string, which passes through a fixed ring, the natural length of the string being equal to the distance between the ring and the fixed point.

(a) If the body receive an impulse it will describe an ellipse round the place it would occupy if suspended freely.

(b) When does this ellipse become a right line?

4. A particle is attached by a straight elastic string to a centre of repulsive force, the intensity of which varies as the distance; the string is at first at its natural length. Find the greatest distance from the centre of force to which the particle will proceed, and the time the string takes to return to its natural length.

5. Two bodies, W and W', hang at rest, being attached to the lower end of a fine elastic string, whose upper end is fixed; supposing one of them, W' to drop off, find the subsequent motion of the other.

Let a be the natural length of the string; b its extension of length for the weight W; c that for the weight W'; then, at the end of any time t, from the commencement of the motion x, the depth of W below the fixed point is given by the equation $x = a + b + c \cos t\sqrt{\frac{g}{b}}$.

6. Two particles, connected by a fine elastic string, are moving in the direction of the line joining them with equal velocities, their distance being the natural length of the string: if the hinder particle be suddenly stopped, find how far the other will move before it begins to return.

SECTION II.—*Central Orbits.*

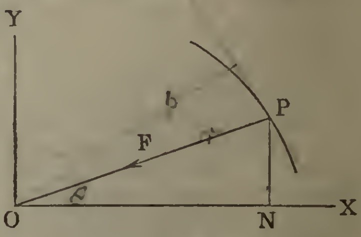
147. **Plane of Orbit.**—If we suppose a particle acted on by a force directed to a fixed centre to be projected in any direction, it is easily seen that its subsequent path will lie in the plane passing through the centre of force and the direction of its projection. For, since the force acts towards the fixed centre, it has no tendency to withdraw the particle from that plane at the first instant, nor at any subsequent instant during the motion; because the motion of the particle at each instant is got by compounding its previous motion with that due to the central force.

We shall accordingly take this plane, called the *plane of the orbit*, as the plane of rectangular coordinate axes; the fixed centre of force being the origin O.

148. **Differential Equations of Motion.**—Suppose the force attractive, and P the position of the attracted particle at the end of any time t.

Let

$$ON=x,\ PN=y,\ OP=r,\ \angle XOP=\theta.$$



Suppose F to represent the *acceleration* due to the attractive force; then, by Art. 67, we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -F\cos\theta = -F\frac{x}{r}\\ \frac{d^2y}{dt^2} &= -F\sin\theta = -F\frac{y}{r}\end{aligned}\right\}. \qquad (1)$$

The complete determination of the motion for any law of force depends on the solution of these simultaneous equations.

In the case of a *repulsive* force it is necessary to change the sign of F.

The path described is evidently always *concave* to the centre of force for *attractive* forces, and *convex* for *repulsive.*

149. **Law of Direct Distance.**—There is one case in which the differential equations can be immediately integrated, viz., when the force varies directly as the distance from the fixed centre.

Let $F = \mu r$; then, for attractive forces, we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} + \mu x = 0\\ \frac{d^2y}{dt^2} + \mu y = 0\end{aligned}\right\}. \qquad (2)$$

The integrals of these equations, by Art. 107, may be written

$$\left.\begin{aligned}x &= A\cos t\sqrt{\mu} + B\sin t\sqrt{\mu}\\ y &= A'\cos t\sqrt{\mu} + B'\sin t\sqrt{\mu}\end{aligned}\right\}. \qquad (3)$$

The arbitrary constants in this, as in all other cases, can be found from knowing the position, velocity, and direction of motion at the first instant.

150. **Equation of Orbit, and Periodic Time.**—If we solve the preceding equations for $\cos t\sqrt{\mu}$ and $\sin t\sqrt{\mu}$, and add the squares of the results, we get

$$(Ay - A'x)^2 + (By - B'x)^2 = (AB' - BA')^2. \qquad (4)$$

This equation represents an ellipse, whose centre is at the centre of force.

Again, if $2\pi + t\sqrt{\mu}$ be substituted for $t\sqrt{\mu}$ in equations (3) the values of x and y remain unaltered; hence, if $(t' - t)\sqrt{\mu} = 2\pi$, the body will occupy the same position at the end of the time t' which it occupied at the time t. Accordingly it makes a complete revolution in its orbit in the time represented by $\frac{2\pi}{\sqrt{\mu}}$.

This is called the *periodic* time, and is the same for all orbits round the same centre of force, since it depends only on μ, the intensity of the central force, *i.e.* the acceleration at the unit of distance, and not on the initial conditions of the motion.

151. **Determination of the Arbitrary Constants.**—Let a, b be the coordinates of the particle at the instant from which the time is reckoned, V the initial velocity, and α the angle which the initial direction of motion makes with the axis of x; then, making $t = 0$ in equations (3), we get $A = a$, $A' = b$.

Again, by differentiation, we have

$$\frac{dx}{dt} = B\sqrt{\mu}\cos t\sqrt{\mu} - A\sqrt{\mu}\sin t\sqrt{\mu},$$

$$\frac{dy}{dt} = B'\sqrt{\mu}\cos t\sqrt{\mu} - A'\sqrt{\mu}\sin t\sqrt{\mu}.$$

Hence $$V\cos\alpha = B\sqrt{\mu}, \quad V\sin\alpha = B'\sqrt{\mu};$$

consequently $$\left.\begin{aligned} x &= a\cos t\sqrt{\mu} + \frac{V\cos\alpha}{\sqrt{\mu}}\sin t\sqrt{\mu} \\ y &= b\cos t\sqrt{\mu} + \frac{V\sin\alpha}{\sqrt{\mu}}\sin t\sqrt{\mu} \end{aligned}\right\}; \qquad (5)$$

thus the position of the particle at any instant is determined.

152. **Repulsive Force.**—Next, if the force be repulsive the equations of motion are

$$\frac{d^2x}{dt^2} = \mu x, \quad \frac{d^2y}{dt^2} = \mu y.$$

Hence, as before,
$$\left.\begin{aligned} x &= A e^{t\sqrt{\mu}} + B e^{-t\sqrt{\mu}} \\ y &= A' e^{t\sqrt{\mu}} + B' e^{-t\sqrt{\mu}} \end{aligned}\right\}. \qquad (6)$$

If we solve for $e^{t\sqrt{\mu}}$ and $e^{-t\sqrt{\mu}}$, and multiply the resulting values, we get
$$(A'x - Ay)(By - B'x) = (A'B - B'A)^2. \qquad (7)$$

This represents a hyperbola, having the lines
$$A'x - Ay = 0, \quad By - B'x = 0$$
for its asymptotes. The constants A, B, A', B' can be easily determined, as in the former case, whenever the initial position, velocity, and direction of motion are given.

Conversely to the preceding Articles, it can be readily shown that if a particle describe a conic under the action of a force directed to its centre, the force varies directly as the distance; and is attractive for an ellipse, and repulsive for a hyperbola.

153. **Several Centres of Force.**—The results arrived at above hold for the motion of a body acted on by any number of centres of force, each varying directly as the distance. For it is readily seen that, in this case, the forces are equivalent to a single force, directed to the centre of mean position of the different centres of force, whose intensity or *absolute* force is equal to the sum of the intensities of the different centres of force (see Minchin's *Statics*, Art. 17).

In like manner, if we suppose each particle of a body to attract according to the law of direct distance, its total attraction is the same as if its entire mass were concentrated at its centre of inertia.

Hence it follows that if two bodies mutually attract, according to this law, they will describe ellipses, in the same periodic time, round their common centre of inertia. The same holds for any number of mutually attracting bodies.

EXAMPLES.

1. Prove that the velocity at any point in a central elliptic orbit varies directly as the diameter drawn parallel to the tangent at the point.

2. In the case of a repulsive force, varying as the distance, find the arbitrary constants, the initial conditions being supposed the same as in Art. 151.

Making $t = 0$ in equations (6), we get $a = A + B, \quad b = A' + B'$.

Again, by differentiation, on making $t = 0$, we get

$$V \cos \alpha = (A - B)\sqrt{\mu}, \quad V \sin \alpha = (A' - B')\sqrt{\mu}.$$

Hence,

$$A = \frac{1}{2}\left(a + \frac{V\cos\alpha}{\sqrt{\mu}}\right), \quad B = \frac{1}{2}\left(a - \frac{V\cos\alpha}{\sqrt{\mu}}\right),$$

$$A' = \frac{1}{2}\left(b + \frac{V\sin\alpha}{\sqrt{\mu}}\right), \quad B' = \frac{1}{2}\left(b - \frac{V\sin\alpha}{\sqrt{\mu}}\right).$$

3. Find the condition that the orbit in the preceding should be an equilateral hyperbola. *Ans.* $V^2 = (a^2 + b^2)\mu$.

4. A particle is acted on by four equal masses, attracting directly as the distance; find its orbit, and show that its periodic time is one-half of that of a particle acted on by one of the masses alone.

5. A particle is attracted to one fixed centre, and repelled by another, of equal intensity, each varying directly at the distance. Find its path.
Ans. A parabola.

6. In the ellipse described freely by a material particle, under the action of a central force varying directly as the distance, determine the relation connecting the eccentric angle of position with the time of passage through any point on the curve.

7. A number of bodies, which describe ellipses about the centre of force as centre in the same periodic time, are projected from a given point with a given velocity in different directions in a plane. Prove that their paths will all touch a fixed ellipse with the given point as focus. *Camb. Math. Trip.* 1875.

8. Being given the centre of force, a point in the orbit, and the velocity and direction of motion at that point, give a geometrical construction for the lengths and positions of the axes major and minor of the orbit.

We now return to the general equations of motion under

CENTRAL FORCES.

154. Equable Description of Areas.—In equations (1) if the first be multiplied by y, and the second by x, we get by subtraction

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0, \text{ or } \frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0.$$

Hence $$x\frac{dy}{dt} - y\frac{dx}{dt} = h,$$

where h is a constant independent of the time.

Again (Art. 105, *Diff. Calc.*), we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt};$$

therefore $$r^2\frac{d\theta}{dt} = h. \tag{8}$$

Hence, if A denote the area described in the time t by the radius vector r drawn to the particle, we have

$$\frac{dA}{dt} = \tfrac{1}{2}\left(r^2\frac{d\theta}{dt}\right) = \tfrac{1}{2}h;$$

therefore $$A = \tfrac{1}{2}(ht). \tag{9}$$

No constant is added since we suppose A and t to vanish together.

If we suppose $t = 1$, we infer that h is *double the area described by the radius vector in the unit of time.*

Conversely, if a particle move in a plane, and describe equal areas in equal times around a fixed point in the plane, then the entire force acting on it at each instant passes through the fixed point (compare Art. 28).

155. **Velocity at any Point.**—Again (Art. 183, *Diff. Calc.*), we have

$$p\frac{ds}{dt} = r^2\frac{d\theta}{dt},$$

where ds denotes the element of the path described in the time dt, and p is the length of the perpendicular from the centre of force on the tangent at the point. Hence

$$p\frac{ds}{dt} = h; \text{ but } \frac{ds}{dt} = v,$$

where v denotes the velocity at the instant; therefore

$$v = \frac{h}{p}. \tag{10}$$

Accordingly the velocity varies inversely as the perpendicular p.

The constant h can be determined from (10) whenever the velocity V, the distance R, and the direction of motion at any point of the path of the particle, are known.

For, let ϕ denote the angle which the direction of motion, at the instant, makes with the radius vector R; then the perpendicular on the tangent $= R \sin \phi$, and hence

$$h = VR \sin \phi. \tag{11}$$

Equation (10) admits of another form; for, squaring, it becomes

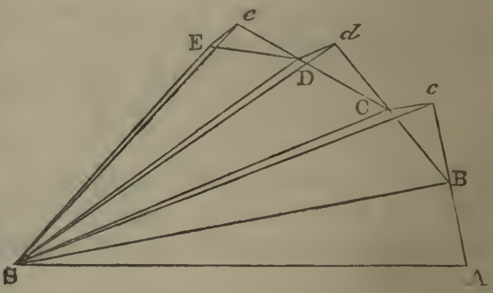
$$v^2 = \frac{h^2}{p^2};$$

therefore

$$v^2 = h^2 \left\{ u^2 + \left(\frac{du}{d\theta} \right)^2 \right\}, \tag{12}$$

where $u = \dfrac{1}{r}$ (*Diff. Calc.*, Art. 183).

156. **Newton's Proof.**—On account of the importance of the preceding results we shall give the method by which the equable description of areas was originally established by Newton.



Let the whole time be divided into a number of equal intervals. Then, supposing no force to act on the body during the first interval, it would describe a right line AB, uniformly, in that interval. Likewise during the next interval, if no force act on it, it would describe the right line Bc, in the direction of, and equal to, AB. But when the body arrives at B, suppose a force directed to S to act on it, with a *single sudden and great impulse*, so as to cause the body to deviate from the right line Bc, and to proceed along the line BC. To find the position of the body at the end of the second interval, we draw from c the line cC parallel to BS (the direction of the force), and meeting BC in C; then the body will be found at C at the end of this interval. Join SC and Sc; then, since

SB and *Cc* are parallel, the triangle *SBC* is equal to *SBc*, and therefore equal to the triangle *SAB*. In like manner *D*, *E*, &c., the positions at the end of the next intervals, can be determined. Also it is obvious that the right lines *AB*, *BC*, *CD*, &c., all lie in the same plane, and the triangles *SCD*, *SDE*, &c., will be each equal to *SAB*.

Therefore equal areas round *S* are described in equal intervals of time; and, *componendo*, the sum of the areas described are proportional to the time of their description.

If now we suppose the number of intervals of time increased, and their length diminished indefinitely, the path described becomes a curved line; the centripetal force by which the body is perpetually deflected from the tangent to the curve will act continuously; and the areas described round *S*, being always proportional to the time of their description, will be so in this case also.

The other results of the preceding Article follow likewise (Newton, Lib. I., Sec. II., Prop. i., *Principia*).

157. **Velocity at any Distance.**—In equations (1) if we multiply the first by $2\,dx$, and the second by $2\,dy$, and add, we get

$$2\frac{d^2x}{dt^2}\,dx + 2\frac{d^2y}{dt^2}\,dy = -\,2F\frac{x\,dx + y\,dy}{r} = -\,2F\,dr.$$

Integrating, we get

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = -\,2\int F\,dr + \text{const.},$$

or

$$v^2 = -\,2\int F\,dr + \text{const.} \tag{13}$$

By aid of this equation, when the law of force is given, the velocity at any point in the orbit can be determined.

Thus, let the acceleration F be any function of the distance represented by $\mu\,\phi'(r)$, then

$$v^2 = -\,2\mu\int\phi'(r)\,dr + \text{const.} = -\,2\mu\phi(r) + \text{const.}$$

Again, let V be the velocity at the distance R, and we get

$$V^2 = -\,2\mu\phi(R) + \text{const.};$$

therefore

$$v^2 - V^2 = \;\; 2\mu\{\phi(R) - \phi(r)\}. \tag{14}$$

For instance, for the law of nature, we have

$$v^2 - V^2 = 2\mu\left(\frac{1}{r} - \frac{1}{R}\right). \tag{15}$$

Hence we see that the velocity at any distance from the centre of force is independent of the path described, and is the same as if the body had been projected, with the initial velocity, directly towards the centre of force (compare Art 128).

Again, if $F = \frac{\mu}{r^n}$, we have

$$v^2 - V^2 = \frac{2\mu}{n-1}\left(\frac{1}{r^{n-1}} - \frac{1}{R^{n-1}}\right). \tag{16}$$

If $V = 0$, when $R = \infty$, *i.e.* if *the velocity at any point in the path is that which the body would acquire in moving from rest from an infinitely great distance* towards the centre of force, we have

$$v^2 = \frac{2\mu}{n-1}\,\frac{1}{r^{n-1}}. \tag{17}$$

For instance, if the force vary as the inverse square of the distance, we have in this case

$$v^2 = \frac{2\mu}{r}. \tag{18}$$

Again, if the force be *repulsive*, and vary directly as the n^{th} power of the distance, we have $F = -\mu r^n$, and (14) becomes

$$v^2 - V^2 = \frac{2\mu}{n+1}\,(r^{n+1} - R^{n+1}). \tag{19}$$

If $V = 0$ when $R = 0$, *i.e.* if *the velocity at any point be the same as that acquired in moving from the centre of force,*

$$v^2 = \frac{2\mu}{n+1}\,r^{n+1}. \tag{20}$$

158. To prove the relation $F = \frac{h^2}{p^3}\frac{dp}{dr}$. (21)

Equation (13) gives, by differentiation,

$$F = -\tfrac{1}{2}\frac{d(v^2)}{dr} = -\tfrac{1}{2}h^2\frac{d}{dr}\left(\frac{1}{p^2}\right) = \frac{h^2}{p^3}\frac{dp}{dr}.$$

This result admits of a useful transformation; for, if γ denote the semichord of curvature drawn through the centre of force, we have

$$\gamma = p\frac{dr}{dp}. \qquad \text{(\textit{Diff. Calc.}, Art. 235.)}$$

Hence the previous equation becomes

$$F = \frac{v^2}{\gamma}. \tag{22}$$

This result can also be readily deduced from the consideration that the centrifugal acceleration, $\frac{v^2}{\rho}$, at any point in the orbit, must be equal and opposite to the component of the central acceleration taken in the normal direction (Arts. 25, 87).

EXAMPLES.

1. Prove that the velocity at any point in a central orbit is the same as that acquired in moving from rest along one-fourth the chord of curvature at the point, under the action of a constant force, equal in intensity to that of the central force at the point.

2. A particle describes a circle freely under the action of a force whose direction is constant: determine the law of force.

Taking the centre of the circle as origin of rectangular axes, the axis of y being parallel to the constant direction of the force, we have

$$\frac{d^2y}{dt^2} = Y, \quad \frac{d^2x}{dt^2} = 0, \quad x^2 + y^2 = a^2;$$

$$\therefore \quad \frac{dx}{dt} = \alpha, \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0;$$

hence $$\frac{dy}{dt} = -\alpha\frac{x}{y};$$

hence
$$Y = \frac{d^2y}{dt^2} = \frac{a}{y^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = -\frac{a^2 a^2}{y^3}.$$

3. Apply equation (21) to find the law of force directed to a focus in an ellipse. In this case we have

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right);$$

$$\therefore \quad \frac{1}{p^3}\frac{dp}{dr} = \frac{a}{b^2}\frac{1}{r^2}; \qquad \text{hence } F = \frac{ah^2}{b^2 r^2}.$$

4. Find the law of force in the curve

$$r^m = a^m \cos m\omega.$$

Here we have (*Diff. Calc.*, Art. 190), $r^{m+1} = a^m p$.

Hence
$$F = \frac{(m+1)h^2 a^{2m}}{r^{2m+3}}.$$

5. Prove that the force under whose action a body P revolves in any orbit about a centre of force S is to the force under whose action the same body P can revolve in the same orbit, in the same time, round another centre of force R, as $SP \cdot RP^2 : SG^3$, where SG is the straight line drawn from S parallel to RP, meeting in G the tangent at P to the orbit. *Principia*, Sect. II., Prop. vii., *Cor.* 3.

159. **To prove the Equation** $\dfrac{d^2u}{d\theta^2} + u = \dfrac{F}{h^2 u^2}$.

In the equation
$$\frac{d(v^2)}{dr} = -2F,$$

if we regard r as a function of θ, we have

$$-2F = \frac{\dfrac{d(v^2)}{d\theta}}{\dfrac{dr}{d\theta}} = -\frac{u^2}{\dfrac{du}{d\theta}}\frac{d(v^2)}{d\theta}.$$

Moreover, from (12), we have

$$\frac{d(v^2)}{d\theta} = 2h^2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right).$$

Substituting in the preceding, we get

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}. \tag{23}$$

This important result can also be proved as follows :—
Substituting $-F$ for P, in equation (9), Art. 28, we get

$$F = -\frac{d^2r}{dt^2} + r\left(\frac{d\theta}{dt}\right)^2;$$

but $$r\left(\frac{d\theta}{dt}\right)^2 = h^2u^3, \text{ by (8)};$$

also $$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = hu^2\frac{dr}{d\theta} = -h\frac{du}{d\theta};$$

$$\therefore \frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d\theta}{dt}\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2};$$

consequently $$F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right).$$

The discussion of central orbits comprises two distinct classes of questions. In the one it is required to find the equation of the orbit when the law of force is known : in the other the orbit described is given, and the law of force, directed to a fixed point, is required.

In the latter case, if the origin be taken at the fixed centre of force, the equation of the orbit can, in general, be expressed in terms of u and θ, from which the value of $\frac{d^2u}{d\theta^2}$ can be determined ; if this be substituted in the equation

$$F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right),$$

the resulting value of F determines the required law of force.

160. **Application to Ellipse.**—For example, to find the law of force which will cause a particle to describe an ellipse round a centre of force situated in one of its foci.

Here the equation of the orbit is

$$u = \frac{1 + e\cos\theta}{L},$$

where L is the semi latus-rectum.

Hence
$$\frac{d^2u}{d\theta^2} = -\frac{e\cos\theta}{L};$$

therefore
$$u + \frac{d^2u}{d\theta^2} = \frac{1}{L},$$

and consequently

$$F = \frac{h^2u^2}{L} = \frac{h^2}{a(1-e^2)}\frac{1}{r^2}. \tag{24}$$

Accordingly the force varies inversely as the square of the distance from the centre of force.

Examples.

Find the law of force, directed to the origin, in the following curves:—

1. $r = ae^{\alpha\theta}$. 2. $u = ae^{\alpha\theta} + be^{-\alpha\theta}$. 3. $r = ae^{\alpha\theta} + be^{-\alpha\theta}$.

Ans. 1. and 2. $\frac{1}{r^3}$. 3. $F = \frac{\mu}{r^2}\left(\frac{1+\alpha^2}{r} - \frac{8\alpha^2ab}{r^3}\right)$.

161. **Case where the Law of Force is Given.**—When the law of force is given, the determination of the orbit depends on the solution of a differential equation; for, if $F = \mu\phi(u)$, equation (23) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\frac{\phi(u)}{u^2}. \tag{25}$$

This equation admits of being completely integrated for a few laws of force only. We shall commence with the most important case, namely the law of nature, for which the attraction varies as the inverse square of the distance.

162. **Law of Inverse Square.**—Let $F = \frac{\mu}{r^2} = \mu u^2$, then the equation becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}.$$

The integral of this, by Art. 108, is

$$u = \frac{\mu}{h^2} + A \cos(\theta - \alpha). \qquad (26)$$

This is the equation of a conic having the centre of force at one of its foci.

The orbit is an ellipse, parabola, or hyperbola, according to the values of the constants A and α. These constants are, as in all other cases, determined from the initial circumstances of the motion.

We commence with the case in which the orbit is an ellipse.

The equation of an ellipse referred to a focus as origin, and to any line drawn through it as prime vector, may be written

$$u = \frac{1}{r} = \frac{1 + e \cos(\theta - \alpha)}{a(1 - e^2)}.$$

Comparing with (26) we get

$$\frac{\mu}{h^2} = \frac{1}{a(1 - e^2)} = \frac{a}{b^2},$$

or

$$h^2 = \mu \frac{b^2}{a} = \mu L. \qquad (27)$$

Hence, in different orbits round the same centre of force, h varies as the square root of the latus rectum.

Again, let T denote the *periodic time*, i.e. the time in which the body makes a complete revolution in the orbit; then, since h represents double the area described in the unit of time, we have

$$h = \frac{\text{double area of ellipse}}{T} = \frac{2\pi ab}{T}.$$

Hence, from (27),

$$\mu = \frac{4\pi^2 a^3}{T^2}. \tag{28}$$

If a second particle be supposed to describe an ellipse round the centre of force, and if the *absolute force* μ be the same in both cases, we have

$$\mu = \frac{4\pi^2 a'^3}{T'^2},$$

where a', T' are the semi-axis and the periodic time in its orbit. Hence, eliminating μ, we get

$$\left(\frac{T}{T'}\right)^2 = \left(\frac{a}{a'}\right)^3. \tag{29}$$

That is, the *squares of the periodic times are to one another in the same ratio as the cubes of the semi-axes major.*

163. The preceding results have been deduced for the motion of a material particle, but they also hold, approximately, for the motion of the centre of inertia of a body of finite dimensions, each of whose elements is attracted towards a fixed centre, provided the dimensions of the body are small in comparison with its distance from the centre of force. For in this case the attractions on the several elementary particles of the body may be, approximately, regarded as a system of equal and parallel accelerations; and, consequently, the motion of the body will (Art. 33) be the same as if it were concentrated at its *centre of inertia.* Also, as already stated in Art. 141, if a sphere consist of homogeneous spherical strata, its entire attraction is the same as if its entire mass were concentrated at its centre. Accordingly, if one such sphere be attracted by another supposed at rest, its centre will describe an ellipse, having the centre of the attracted sphere for a focus.

164. **Kepler's Laws.**—By comparing the results of a large series of observations of the planets, chiefly of Mars, made by Tycho Brahe, Kepler arrived at the following laws concerning the planetary orbits :—

(1) That the right line drawn from the Sun to any planet describes equal areas in equal times.

(2) That the orbits are ellipses, having the Sun in a focus.

(3) That the squares of the periodic times for any two planets are to each other in the same proportion as the cubes of their mean distances from the Sun.

From the first of these laws Newton deduced (Art. 154) that each of the planets is kept in its orbit by the action of a central force directed to the Sun.

From the second he proved that the attractive force for each planet, in its different positions, varies as the inverse square of the distance from the Sun (Art. 160).

For Newton's demonstrations the student is referred to the *Principia*, Lib. I., Sect. III., Prop. xi.

From the third law he deduced that the absolute force (μ) is the same for all the planets (Art. 162) ; and hence that it is one and the same force, directed to the Sun, by which all the planets are retained in their orbits. These laws are only approximate when we take account of the mutual actions of the planets on each other and on the Sun.

From the foregoing we infer that the results arrived at for the motion of a particle, for the law of inverse square of the distance, are applicable, approximately, to the planetary motions. It has also been verified by observation that a satellite belonging to any planet revolves round it according to the same laws that the planets revolve round the Sun.

165. **Law of Gravitation.**—We have in the last Article given a brief outline of the process by which Newton established the great fundamental law of attraction of matter, which we refer to as the law of nature, and which may be stated as follows :—*Every particle of matter in the solar system, consisting of the Sun, the planets, comets, &c., exercises on every other particle an attractive force, which varies directly as the product of the masses of the particles, and inversely as the square of their mutual distance.*

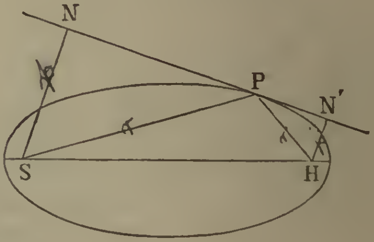
We assume that this is a general property of matter, and applies to all matter wherever existing in the universe. This assumption has been verified by observations on the motion of the double stars.

166. **Expression for Velocity at any point in a Focal Orbit.**—We commence with an elliptic orbit.

In this case we have

$$v^2 = \frac{h^2}{p^2} = \frac{\mu}{a}\frac{b^2}{p^2}, \text{ by (27)},$$

where $p = SN$.



Again, let H be the second focus of the orbit, SN, HN' perpendiculars on the tangent at P, the position of the particle.

Suppose $HN' = p'$, $HP = r'$; then, from well-known elementary properties of the ellipse, we have

$$r + r' = 2a, \quad pp' = b^2, \quad \frac{p'}{p} = \frac{r'}{r}.$$

Hence $$v^2 = \frac{\mu}{a}\frac{pp'}{p^2} = \frac{\mu}{a}\frac{p'}{p} = \frac{\mu}{a}\frac{r'}{r} = \frac{\mu}{a}\left(\frac{2a-r}{r}\right);$$

therefore $$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}. \qquad (30)$$

In the parabola a becomes infinite, and we have

$$v^2 = \frac{2\mu}{r}, \qquad (31)$$

a result which can be readily established independently.

In the case of a hyperbolic path we have $r' = 2a + r$, and the formula becomes

$$v^2 = \frac{2\mu}{r} + \frac{\mu}{a}. \qquad (32)$$

Hence we infer that if a body be projected with a velocity V, at a distance R from the centre of force, the orbit described will be an ellipse, parabola, or hyperbola, according as

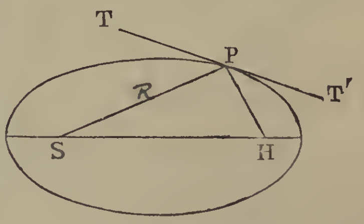
$$V^2 \text{ is } < = \text{ or } > \frac{2\mu}{R}.$$

This result may be exhibited in another form by aid of equation (18), as follows:—

The velocity at any point in an ellipse is less, in a parabola equal to, and in a hyperbola greater than, the velocity which the body would acquire in moving to the point from an infinitely great distance, under the action of the central force.

167. **Construction of Orbit.**—The preceding equation shows how to construct the orbit when we are given the absolute force, the initial velocity, position, and direction of motion. For, suppose P the initial position, PT the direction of motion, and S the centre of force: let V = velocity of projection, $SP = R$; then (1) if $V^2 < \frac{2\mu}{R}$ the orbit is an ellipse whose semi-axis a is given by the equation

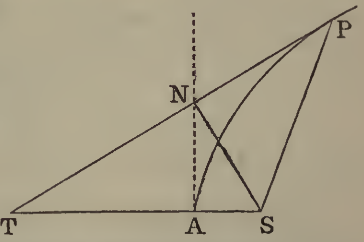
$$\frac{1}{a} = \frac{2}{R} - \frac{V^2}{\mu}.$$



Again, draw PH, making the angle $T'PH = \angle SPT$, then the second focus H lies on this line, and its position H is found by taking $PH = 2a - R$. Consequently, as the two foci and the axis major are known, the ellipse is completely determined.

(2) When $\frac{2\mu}{R} = V^2$ the orbit is a parabola, which can be easily determined by drawing SN perpendicular to the direction of motion at P, inflecting $ST = SP$, and dropping NA perpendicular to ST.

The parabola described with S for focus, and A for vertex, will be the required orbit.



(3) When $V^2 > \frac{2\mu}{R}$ the orbit is a hyperbola, whose semi-axis a is given by the equation

$$\frac{1}{a} = \frac{V^2}{\mu} - \frac{2}{R}.$$

The second focus, H, can be easily constructed, as in the first case, but lies on the opposite side of the direction of motion from the centre of force S.

Again, as the value of the semi-axis a is independent of the direction of projection, we infer that if a number of bodies be projected from a point with the same velocity, in different directions, and be attracted by a common centre of force, the mean distances, and consequently the periodic times, will be the same for all the orbits.

It may be remarked that the orbit will be a circle provided the angle SPT is right, and $V^2 = \frac{\mu}{R}$ (compare Art. 89).

The formulæ in this and the preceding Article are of importance in the discussion of focal orbits. We add a few elementary applications.

Examples.

1. Calculate, approximately, the periodic time of a planet if its mean distance from the Sun is double that of the Earth. *Ans.* 1033 days.

2. If a body be projected with a given velocity about a centre of force which varies as the inverse square of distance, find the locus of the centre of the orbit described.

Here, since the locus of the empty focus is a circle, the locus of the centre is also a circle.

3. In the same case show that the length of the axis-minor varies directly as the perpendicular drawn from the centre of force to the direction of projection.

Since r and r' are each constant, p is to p' in a constant ratio; consequently b varies as p.

4. Show that there are two directions in which a body may be projected from a given point A, with a given velocity V, so as to pass through another given point B.

Since the axis-major $2a$ is given, the position of the second focus is determined by the intersection of two circles, with A and B for centres. Hence there are two solutions—one for each point of intersection of the circles.

5. Prove that the time of describing an arc of a parabolic orbit, bounded by a focal chord of length c, varies as $c^{\frac{3}{2}}$.

168. **Effect of a Sudden Change in Absolute Force.**—A body is revolving in a focal orbit; if when it arrives at any position the absolute force μ be suddenly altered, to determine the subsequent path.

Let R and V represent the distance and velocity at the instant in question, and let μ' be the new value of the absolute force, and a' the semi-axis major of the new orbit; then, as the velocity receives no sudden or instantaneous change, we have, by (30),

$$\frac{2\mu}{R} - \frac{\mu}{a} = \frac{2\mu'}{R} - \frac{\mu'}{a'}. \tag{33}$$

The value of a', and consequently the position of the new orbit, can be immediately determined from this equation.

For example, suppose the original orbit a parabola, and the central force suddenly doubled in intensity.

Here $\mu' = 2\mu$, and our equation becomes

$$\frac{2\mu}{R} = \frac{4\mu}{R} - \frac{2\mu}{a'}:$$

hence $a' = R$; and, consequently, the new orbit is an ellipse having the extremity of its axis major at the point.

If the change in μ be very small, and represented by $\Delta\mu$, and the corresponding change in a by Δa, it is plain that we have

$$\Delta a = -\frac{a^2}{\mu}\Delta\mu\left\{\frac{2}{R} - \frac{1}{a}\right\}. \tag{34}$$

Hence if the central force (or the attracting mass) be increased slightly, the axis major will be diminished; also, if the force be diminished the axis major is increased.

The corresponding change in the periodic time is readily found; for, by (28), we have

$$2\log T + \log\mu = 2\log 2\pi + 3\log a;$$

hence
$$\frac{2\Delta T}{T} = \frac{3\Delta a}{a} - \frac{\Delta\mu}{\mu};$$

therefore
$$\frac{\Delta T}{T} = -\frac{\Delta\mu}{\mu}\left(\frac{3a}{R} - 1\right). \tag{35}$$

Again, if the centre of force be supposed suddenly transferred to a new position, the subsequent path can be readily constructed, as in Art. 167.

EXAMPLES.

1. A number of bodies are projected from a point with the same velocity, but in different directions; prove that the centres of their orbits are situated on the surface of a sphere.

2. A body is describing a circle under a central force in its centre; if the force be suddenly reduced to one-half, find the subsequent path of the body.

Ans. a parabola.

3. In the same case if the central force be suddenly increased in the ratio of $m:1$, find the eccentricity of the subsequent path.

Ans. $\dfrac{m-1}{m}$.

4. Two equal perfectly elastic particles describe the same ellipse in the same period, in opposite directions, one about each focus; prove that the major axis of the orbit is a harmonic mean between those of the orbits they will describe after impact.

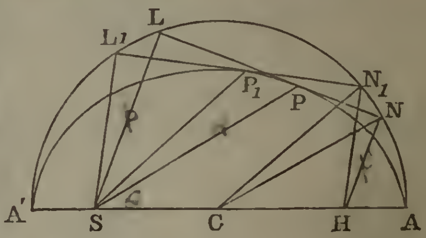
This result follows immediately since the *vis viva* is the same after collision as before (*see* Art. 82).

5. Prove that there are two initial directions for the projection of a particle with a given velocity, so that the axis major of its orbit may coincide in direction with a given line.

6. If, when the Earth is at an end of the minor axis of its elliptic orbit, a meteor were to fall into the Sun, whose mass is the m^{th} part of that of the Sun; find the resulting change in the Ecarth's mean distance, and also in the length of the year.

Ans. $\Delta a = -\dfrac{a}{m}$, $\Delta T = -\dfrac{2T}{m}$.

169. **Application of Method of Hodograph.**—The method of the hodograph (Art. 26) furnishes a simple mode of determining the law of force in a focal ellipse. For, since the velocity at any point P varies inversely as the perpendicular SL, it varies directly as the perpendicular HN drawn from the second focus; since $SL \times HN = b^2$.



Consequently the hodograph is similar to the locus of N, when turned through a right angle. But the semicircle described on the axis major as diameter passes through N, consequently the hodograph is a circle.

Again, to find the law of force, let P_1 denote the position of the moveable at the end of an indefinitely small time Δt, and N_1 the corresponding position of N; then (Art. 26) $\frac{NN_1}{\Delta t}$ is proportional to the central attractive force.

Join the centre C to N and to N_1; then, by an elementary property of the ellipse, CN is parallel to SP, and CN_1 to SP_1.

Let $SP = r$, $\angle\, CSP = \theta$, $SL = p$, $HN = p'$;

then
$$\angle\, NCN_1 = \angle\, PSP_1 = \Delta\theta.$$

Also (by 8),
$$\frac{NN_1}{\Delta t} = a\frac{\Delta\theta}{\Delta t} = \frac{ah}{r^2}.$$

Hence the force varies inversely as the square of the distance.

Again, since $v = \frac{h}{p} = \frac{h}{b^2}p'$, we have

$$F = \frac{h}{b^2}\frac{NN_1}{\Delta t} = \frac{h^2 a}{b^2}\cdot\frac{1}{r^2}.$$

Consequently, if μ represent the absolute force, *i.e.* the force at unit of distance, we get

$$\mu = \frac{h^2 a}{b^2},$$

as in (27).

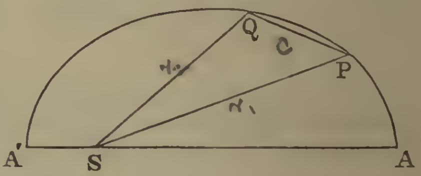
Again, since the velocity at P is proportional, and perpendicular to HN; and CN, CH are constants, it follows that the velocity at P can be resolved into two *constant velocities—one perpendicular to the radius vector, the other to the axis major*. Also, since the velocity at P is represented by $\frac{h}{b^2}HN$, the component velocity perpendicular to SP is represented by $\frac{ha}{b^2}$, and that perpendicular to the axis major by $\frac{hae}{b^2}$: i.e. by $\frac{\mu}{h}$ and $\frac{\mu}{h}e$, or by $\sqrt{\frac{\mu}{L}}$ and $\sqrt{\frac{\mu}{L}}\,e$, respectively.

EXAMPLE.

A particle moving in an ellipse under the action of a force directed to a focus has a small velocity $n\frac{\mu}{h}$ impressed on it in the direction of the focus; find the corresponding changes in the eccentricity, and position of the apse.

170. **Lambert's Theorem.**—In Art. 140, *Int. Calc.*, it has been shown that the area of the elliptic sector PSQ is represented by

$$\tfrac{1}{2}ab\,\{\phi - \phi' - (\sin\phi - \sin\varphi')\},$$



where ϕ and ϕ' are given by the equations

$$\sin\tfrac{1}{2}\phi = \tfrac{1}{2}\left(\frac{r_1 + r_2 + c}{a}\right)^{\frac{1}{2}},\ \sin\tfrac{1}{2}\phi' = \tfrac{1}{2}\left(\frac{r_1 + r_2 - c}{a}\right)^{\frac{1}{2}};$$

in which $SP = r_1$, $SQ = r_2$, and $PQ = c$.

Accordingly, if t represent the time of describing the arc PQ, we have

$$t = \frac{2\text{ area } PSQ}{h} = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\{\phi - \phi' - (\sin\phi - \sin\phi')\}. \qquad (36)$$

This shows that the time of moving from any point P to any point Q can be expressed in terms of the sides of the triangle SPQ, and of the axis major of the orbit.

Again, if we regard a as becoming infinitely great in (36), we get for t, the time of moving from P to Q in a parabolic orbit,

$$t = \frac{1}{6\sqrt{\mu}}\{(r_1 + r_2 + c)^{\frac{3}{2}} - (r_1 + r_2 - c)^{\frac{3}{2}}\}. \qquad (37)$$

For in this case we may substitute $\frac{1}{6}\left(\frac{r_1 + r_2 + c}{a}\right)^{\frac{3}{2}}$ for $\phi - \sin\phi$, and $\frac{1}{6}\left(\frac{r_1 + r_2 - c}{a}\right)^{\frac{3}{2}}$ for $\phi' - \sin\phi'$.

EXAMPLES.

1. A comet, describing a parabolic orbit, being supposed to cross the path of the Earth; determine the points of ingress and egress for which the time the comet continues within the Earth's orbit is a maximum.

Ans. The extremities of the axis major.

2. Find an expression for the time in the preceding question.

Ans. $\frac{2E}{3\pi}$, where E represents the length of the year.

3. Two planets, describing elliptic orbits in a common period round the Sun, being supposed to pass in every revolution through two common points; prove that the intervals between the times of their passage through the points are equal.

171. **Modification when Mutual Attraction is taken account of.**—The preceding investigations are based on the assumption that the centre of force is fixed; accordingly they can be applied to the planetary motions only on that hypothesis. However, from the principle of the equality of action and reaction, each of the planets exerts on the Sun an equal and opposite attractive force to that which the Sun exerts on it. We proceed to consider how far our results must be modified when this is taken into account.

We have seen, in Art. 13, that the relative motion of two bodies is unaltered if equal and parallel velocities be given to both. We accordingly suppose an acceleration applied at each instant to the Sun, equal and opposite to that which the planet exerts on it; and an equal and parallel acceleration applied to the planet. This assumption will not alter their relative positions, while it reduces the position of the Sun to one of relative rest. Consequently the *relative motion* of the planet takes place in the same manner as if the Sun were a fixed centre of force, and the planet at each instant were acted on by the sum of the accelerations that the Sun exerts on it, and that it exerts on the Sun; since these accelerations take place in opposite directions along the same right line.

Again, let S and P denote the masses of the Sun and planet respectively: then their attractions (being proportional to their masses) may be represented by $f\frac{S}{r^2}$ and $f\frac{P}{r^2}$, where r represents their mutual distance.

Accordingly the total acceleration on the planet towards the Sun, considered as a fixed centre, is represented by

$$f\,\frac{(S+P)}{r^2}.$$

Consequently in our preceding investigations we must regard the absolute force, μ, as proportional to $S+P$ instead of S; and we may, by proper assumption of units, take

$$\mu = f(S+P).$$

172. **Modification in Kepler's Third Law.**—From what has been just established it follows that Kepler's third law is only approximate. To determine a more exact result we must substitute $f(S+P)$ instead of μ, in (28), for one planet, and $f(S+P')$ in the corresponding formula for the other planet, when we have, by division,

$$\frac{S+P}{S+P'} = \left(\frac{a}{a'}\right)^3 \left(\frac{T'}{T}\right)^2. \qquad (38)$$

As observation shows that Kepler's third law is very nearly exact for all the planets, we conclude that the mass of the Sun is very great in comparison with that of any of the planets. In fact the mass of Jupiter, which is the largest of them, is less than a thousandth part of that of the Sun.

This conclusion will appear more clearly from the following method of comparing the mass of the Sun with that of a planet where the planet has a satellite:—

173. **Comparison of Masses of Sun and Planet.**—Let Σ denote the mass of the satellite, δ its distance from the planet, t its periodic time; then, since the satellite revolves round the planet we have, as in last Article,

$$\frac{P+\Sigma}{S+P} = \left(\frac{\delta}{a}\right)^3 \left(\frac{T}{t}\right)^2. \qquad (39)$$

When the calculations are made it is found that in all cases $\left(\frac{\delta}{a}\right)^3 \left(\frac{T}{t}\right)^2$ is a very small fraction; and hence also $\frac{P}{S}$.

If Σ be supposed very small in comparison with P, as P in comparison with S, we can, by (39), obtain the ratio of the planet's mass to that of the Sun, approximately.

174. **Mass of Sun.**—When applied to the Earth and its satellite the Moon, the preceding formula gives a means of comparing the mass of the Sun with that of the Earth.

Let E, M, represent the masses of the Earth, and the Moon, r their distance, then equation (39) becomes

$$\frac{E+M}{S+E} = \left(\frac{r}{a}\right)^3 \left(\frac{T}{t}\right)^2.$$

Now, as a rough approximation, we assume $\frac{r}{a} = \frac{1}{400}$, *i.e.* that the Sun's distance from us is 400 times that of the Moon. Also we take $\frac{T}{t} = 13{\cdot}4$, or that the year is, approximately, 13·4 times the periodic time of the Moon.

This gives

$$\frac{S+E}{E+M} = \frac{64{,}000{,}000}{179{\cdot}56} = 356{,}420 \text{ approximately.}$$

Moreover, as determined by tidal calculations, $M = \frac{E}{75}$; hence we get

$$\frac{S}{E} = 361{,}473.$$

This result represents very closely the ratio of the Sun's and Earth's mass as determined by more exact investigations.

The foregoing calculation shows the enormous mass of the Sun in comparison with that of the Earth. In like manner the relative masses of Jupiter, Saturn, and other planets which have satellites can be found, approximately.

175. **Mean Density of Sun.**—The ratio of the mean density of the Sun to that of the Earth can be determined, as follows:—

From (39) we have, approximately,

$$\frac{S}{E} = \left(\frac{a}{r}\right)^3 \left(\frac{t}{T}\right)^2.$$

Again, let ρ, ρ_1 denote the radii of the Sun and Earth, and σ the ratio of their mean densities; then, assuming them spherical bodies, we have

$$\frac{S}{E} = \sigma\left(\frac{\rho}{\rho_1}\right)^3.$$

Hence
$$\sigma = \left(\frac{a}{\rho}\right)^3\left(\frac{\rho_1}{r}\right)^3\left(\frac{t}{T}\right)^2,$$

or
$$\sigma = \left(\frac{P}{a}\right)^3\left(\frac{t}{T}\right)^2, \qquad (40)$$

where a denotes the Sun's *mean apparent semi-diameter*, and P the Moon's *mean horizontal parallax.*

If we substitute 16′ for a and 57′ for P, and take $\frac{t}{T}$ as before, we get $\sigma = 0{\cdot}23$, *i.e.* the Sun's mean density is about one-fourth that of the Earth.

It should be observed that this result does not require a knowledge of the Sun's distance; and, as the constants in (40) can be obtained with great accuracy, the ratio of the mean densities of the Sun and Earth can be determined with great precision.

176. **Planetary Perturbations.**—The previous deductions respecting the planetary motions are only approximate for another and a more important reason, namely, that in them we have neglected the mutual actions of the planets on each other.

However, since the Sun's mass is very great in comparison with that of all of the planets, their attractions on any member of the solar system may be regarded as small *disturbing forces*, and the planetary orbits as approximately ellipses.

The usual method of treatment, accordingly, is to regard each planet as moving in an ellipse, in which the elements* are subject to very slow changes, arising from the *perturbations* or disturbing effects of the other planets.

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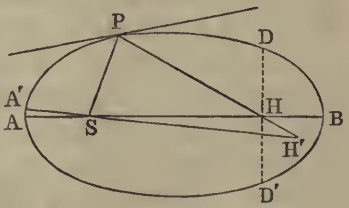
* The elements by which a planet's path is determined are—(1) its mean distance from the Sun; (2) its eccentricity; (3) the longitude of its perihelion; (4) the inclination of its plane to a fixed plane; (5) the angle which the intersection of these planes makes with a fixed line; (6) its epoch, or the instant of the planet's being in perihelion.

In this manner the problem has been discussed by Lagrange, Laplace, and other great writers on Physical Astronomy. We shall not enter into this discussion, as it is beyond the limits contemplated in this treatise. There is, however, one mode of considering the effects of a disturbing force, which may be here introduced. This consists in supposing the disturbing force resolved into two components*, one along the tangent, the other along the normal to the orbit, and in treating their effects separately.

177. **Tangential Disturbing Force.**—Suppose P the position of a planet, moving in the ellipse BPA, in which S and H are the foci; then, since a tangential disturbing force alters the velocity, but produces no effect on the direction of motion, it is easy to find the corresponding changes in the *elements* of the path. For the new position, H', of the second focus will still lie on the line PH.



Again, if v denote the velocity at P, we have, as before,

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}.$$

When the change in v^2, caused by the tangential disturbing force, is known, the corresponding change in a can be found; and hence the position of H', and consequently that of the new axis major.

Thus if δv be the small change in v, due to the disturbing force, we have

$$2v\delta v = \frac{\mu\delta a}{a^2};$$

$$\therefore \quad \delta a = \frac{2a^2}{\mu} v\delta v; \qquad \therefore \quad HH' = 2\delta a = \frac{4a^2}{\mu} v\delta v. \qquad (41)$$

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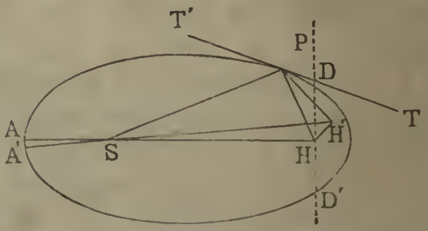
* There is in general a third component, perpendicular to the plane of the orbit. It is not proposed to consider the effects of this component here. This method of treating the disturbing forces is discussed in a masterly and lucid manner by Sir John Herschel, in his *Outlines of Astronomy*, ch. 12 and 13

If the tangential force act in the direction of the motion, and consequently increase the velocity, a will also be increased, and the perihelion A' will consequently move towards P.

Again, the eccentricity e will be increased when SH' is greater than SH, i.e. when P is between the perihelion A and the extremity of the *latus-rectum* drawn through H.

178. **Normal Disturbing Force.**—Next, if a normal disturbing force act inwardly, at P, it does not alter the velocity, but it changes the direction of motion, through a small angle $\delta\phi$. As the velocity is unchanged, the length of the semi-axis major a is unaltered, while the angle SPT is altered by the quantity $\delta\phi$. Therefore the angle HPH', between PH and the corresponding line PH' in the new orbit, is $2\delta\phi$; also $PH = PH'$. In this manner the position of H' is found when the angle $\delta\phi$ is known. Again, join SH', and produce it at both sides, then the line $A'B'$ will represent the direction of the axis major of the new orbit.



Through H draw DD' perpendicular to SH. The points D and D' are called the *quadratures* of the orbit. When P lies between D and A, the perihelion, the line AB, called *the line of apsides* (*see* next Article), moves in the same direction as the planet, and is said to *advance.* The eccentricity increases at the same time. If the planet be between aphelion B and D, the eccentricity continues to increase, and the line of apsides recedes.

Again, in moving from A to D', the *disturbing force still acting inwards*, it is easily seen that the line of apsides *advances*, and the eccentricity diminishes. Hence, in the motion from quadrature to quadrature, through perihelion, the apse continually advances, in the case of a normal *disturbing* force acting inwards; the eccentricity increases during the first half of the motion, and diminishes during the second.

The contrary effects have place for a normal disturbing force acting *outwards.*

In like manner in the motion from quadrature to quadrature through *aphelion*, the apse recedes; the eccentricity increases during the first half and diminishes during the second.

179. **Apsides.**—A position for which the moving body is at a maximum or a minimum distance from the centre of force is called an *apse*. The corresponding distance from the centre of force is called an *apsidal distance*, and the line joining the centre of force to an apse is called an *apsidal line*.

Since r, and consequently u, attains a maximum or a minimum value at an apse, we have at such a point

$$\frac{du}{d\theta} = 0.$$

It is easily seen that the orbit is symmetrical at both sides of an apse, provided the force is a function of the distance only. For, if a particle be supposed projected from a point A in a direction perpendicular to the line OA drawn to the centre of force, it is obvious that for the same velocity of projection we must have exactly similar paths, whether it be projected in any given direction or in that exactly opposite. Moreover, if the velocity were reversed at any point, the body would proceed to describe the same orbit, but in an opposite direction. From these considerations it follows that the central orbit must be symmetrical at both sides of an apse, since at that point the motion is perpendicular to the central radius vector.

180. **An Orbit can have but two Apsidal Distances.**—For, suppose A and B to be two apsides, and the body to move from A to B, then after passing B it will, by the preceding Article, describe a curve similar to BA; and so on. Hence the apsides are constantly repeated, and the angle between two consecutive apsidal distances is the same for all positions of the orbit. This angle is called the *apsidal angle* of the orbit. It is plain that a central orbit cannot be a closed curve unless the apsidal angle is commensurable with a right angle.

181. **Equation for Determination of Apsides.**—Let $F = \mu\phi(u)$, then we have, by (13),

$$v^2 = 2\mu \int \frac{\phi(u)}{u^2}\, du + C$$

where the value of C is determined by the initial conditions;

therefore $$h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = 2\mu\int\frac{\phi(u)}{u^2}\,du + C. \tag{42}$$

Hence, as $\dfrac{du}{d\theta} = 0$ at an apse, the equation for determining the apsidal distances is

$$h^2u^2 = 2\mu\int\frac{\phi(u)}{u^2}\,du + C. \tag{43}$$

If we suppose $F = \mu u^n$, equation (42) becomes

$$h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \frac{2\mu}{(n-1)}u^{n-1} + C, \tag{44}$$

and the equation for the apsides

$$h^2u^2 = \frac{2\mu}{n-1}u^{n-1} + C. \tag{45}$$

The form of the latter equation shows that it cannot have more than two positive roots, which therefore correspond to the two apsidal distances.

For example, let the force consist of two parts, one varying as the inverse square of the distance, the other as the inverse cube, or

$$F = \mu u^2 + \mu' u^3. \tag{46}$$

then $$h^2u^2 = 2\mu u + \mu' u^2 + C.$$

Accordingly the apsidal distances are in this case determined by a quadratic equation. If $\mu = 0$, there is but one apsidal distance.

182. **Case of Velocity due to an Infinite Distance.**—The integration of equation (44) in a finite form is in general impossible; there is, however, one case in which the equation of the orbit can be readily determined, viz., when the velocity at any point is that acquired in moving from an infinite distance under the action of the central force.

For we have, in this case, by (17), $v^2 = \frac{2\mu}{n-1} u^{n-1}$;

therefore $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{(n-1)h^2} u^{n-1}. \qquad (47)$$

Hence $\frac{du}{d\theta} = u\sqrt{au^{n-3} - 1}$, writing a instead of $\frac{2\mu}{(n-1)h^2}$

therefore $$\theta = \int \frac{du}{u\sqrt{au^{n-3} - 1}}.$$

To integrate this, let $au^{n-3} = \frac{1}{z^2}$, then $\frac{du}{u} = -\frac{2}{n-3}\frac{dz}{z}$;

and $\int \frac{du}{u\sqrt{au^{n-3} - 1}}$ becomes $-\frac{2}{n-3}\int \frac{dz}{\sqrt{1-z^2}}$

$$= \frac{2}{n-3}\cos^{-1} z + \text{const};$$

therefore $\theta + \beta = \frac{2}{n-3}\cos^{-1} z$, or $z = \cos\frac{n-3}{2}(\theta + \beta)$,

where β is an arbitrary constant.

Hence we get

$$r^{\frac{n-3}{2}} = \frac{1}{h}\sqrt{\frac{2\mu}{n-1}} \cos\frac{n-3}{2}(\theta + \beta). \qquad (48)$$

If a denote the apsidal distance, and θ be measured from the apsidal line, the preceding may be written

$$r^{\frac{n-3}{2}} = a^{\frac{n-3}{2}} \cos\frac{n-3}{2}\theta. \qquad (49)$$

This is the polar equation of the orbit.

For example, when $n = 2$, we get the parabola

$$r^{\frac{1}{2}} \cos\tfrac{1}{2}\theta = a^{\frac{1}{2}}.$$

Again, when $n = 5$, it becomes

$$r = a \cos \theta;$$

a circle having its centre on the circumference.

For $n = 7$ we get the lemniscate

$$r^2 = a^2 \cos 2\theta,$$

and so on.

Equation (49) fails when $n = 3$; in this case, however (47) becomes

$$\frac{du}{d\theta} = u\sqrt{\frac{\mu}{h^2} - 1},$$

which gives $$\theta\sqrt{\frac{\mu}{h^2} - 1} = \log u + \text{const.},$$

or $$u = \beta e^{\kappa\theta}, \text{ where } \kappa = \sqrt{\frac{\mu}{h^2} - 1}.$$

This is the equation of a logarithmic spiral.

183. **Approximately Circular Orbits.**—If the orbit described round a centre of force be nearly a circle, its equation can be found approximately, as follows:—

Assume $F = \mu u^2 f(u)$, then equation (23) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} f(u).$$

If the orbit were an exact circle we should have

$$u = a, \text{ and } \frac{d^2u}{d\theta^2} = 0;$$

therefore a must satisfy the equation

$$a = \frac{\mu}{h^2} f(a). \qquad (50)$$

When the orbit is approximately circular we may assume $u = a + z$, where z is always very small.

Hence $$\frac{d^2z}{d\theta^2} + a + z = \frac{\mu}{h^2} f(a + z),$$

or $$\frac{d^2z}{d\theta^2} + a + z = \frac{\mu}{h^2} \{f(a) + zf'(a)\}.$$

By (50) this becomes, neglecting z^2 and higher powers of z,

$$\frac{d^2z}{d\theta^2} + z\left(1 - \frac{\mu}{h^2} f'(a)\right) = 0,$$

or, substituting $\frac{a}{f(a)}$ for $\frac{\mu}{h^2}$,

$$\frac{d^2z}{d\theta^2} + z\left\{1 - \frac{af'(a)}{f(a)}\right\} = 0.$$

If $k = 1 - \frac{af'(a)}{f(a)}$, this becomes

$$\frac{d^2z}{d\theta^2} + kz = 0. \tag{51}$$

When k is positive, the integral of this, by Art. 107, is

$$z = c \cos(\theta\sqrt{k} + \alpha),$$

or

$$u = a + c\cos(\theta\sqrt{k} + \alpha), \tag{52}$$

when c and α are arbitrary constants.

The greatest value of u is $a + c$; consequently, in order that the orbit should be approximately circular, it is necessary that c should be very small in comparison with a.

Again, supposing c positive, the greatest value of u is when $\theta\sqrt{k} + \alpha = 0$, and the least when $\theta\sqrt{k} + \alpha = \pi$; consequently the apsidal angle is

$$\frac{\pi}{\sqrt{k}} \text{ or } \frac{\pi}{\sqrt{1 - a\dfrac{f'(a)}{f(a)}}}.$$

If k be negative, *i. e.* if $\frac{af'(a)}{f(a)} > 1$, the integral of (51) is of the form

$$z = Ae^{\theta\sqrt{k}} + Be^{-\theta\sqrt{k}},$$

and therefore z would either increase or diminish indefinitely with θ; and accordingly the orbit cannot be approximately circular in that case.

The value of k depends on the law of force; for example, if the force vary inversely as the n^{th} power of the distance, then

$$f(u) = \mu u^{n-2}, \quad \text{and} \quad \frac{af'(a)}{f(a)} = n - 2.$$

Accordingly, in this case $k = 3 - n$.

Hence a *nearly circular orbit, having the centre of force in the centre, is impossible for laws of force which vary inversely as a higher power than the cube of the distance.*

When n is less than 3 the angle between the apsides is

$$\frac{\pi}{\sqrt{3-n}}$$

For instance, if $n = 2$, the angle is π; which agrees with what has been already proved, as the orbit is a focal conic in this case.

Again, if $n = -1$, the angle is $\frac{1}{2}\pi$, as it ought to be, since the orbit is a central ellipse.

184. **Moveable Orbits.**—If a central orbit be made to move in its own plane, with an angular velocity proportional at each instant to that of the radius vector in the orbit; to prove—(1) that the new orbit is also a central orbit; (2) that the difference between the forces in the two orbits varies inversely as the cube of the distance from the centre of force. (Newton, *Principia*, lib. i., sect. 9.)

In a central orbit we have, in general,

$$r^2\frac{d\theta}{dt} = h, \text{ and } \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P.$$

If now we make $\theta = k\theta'$, where k is constant, the former equation gives

$$r^2\frac{d\theta'}{dt} = \frac{h}{k} = h' \text{(suppose)}.$$

This shows that the point describes equal areas in equal times around the origin; accordingly the new orbit is also a central orbit.

Again, the second equation may be written

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta'}{dt}\right)^2 = P + (k^2 - 1)\,r\left(\frac{d\theta'}{dt}\right)^2$$

$$= P + \frac{h^2 - h'^2}{r^3} = P', \text{ suppose};$$

hence $P' - P = \dfrac{h^2 - h'^2}{r^3}$: showing that the difference between the forces in the fixed and moveable orbits varies as $\dfrac{1}{r^3}$.

Hence, from any central orbit we can get another, called by Newton a revolving orbit; and the equation of the revolving orbit, in polar coordinates, is derived from that of the original by substituting $k\theta$ for θ; where the constant k is determined from the initial conditions.

For example, when $F = \dfrac{\mu}{r^2}$, we get a focal conic, whose equation is of the form

$$r = \frac{A}{1 + e\cos(\theta - a)};$$

hence if $F = \dfrac{\mu}{r^2} + \dfrac{\mu'}{r^3}$, the equation of the orbit is of the form

$$r = \frac{A}{1 + e\cos(k\theta - a)}.$$

The apsidal angle in the new orbit is equal to that in the original orbit divided by k, as is readily seen. Newton applied this method to the determination of the apsidal angle in the lunar orbit. His investigation is beyond the limits proposed in the present treatise. Moreover the progression of the Moon's apse, thus determined by Newton, is but half its true amount.

EXAMPLES.

1. Find the law of force in a circle when the centre of force is situated on its circumference *Ans.* $\frac{1}{r^5}$.

2. Investigate the motion of a body which is acted on by several centres of force varying directly as the distance; and show how to construct the position of the centre of the orbit.

3. In the same case find the condition that the orbit should be a parabola.

4. Assuming that the law of force in a focal conic is that of the inverse square of the distance, show that the converse theorem can be immediately established, viz., that a particle attracted by a centre of force, varying according to that law, will describe a conic, having the centre of force in one of its foci.

5. A semi-ellipse being supposed freely described by a particle under the action of a force parallel to its axis of figure: determine the requisite law of force, with the velocity of the particle on reaching or leaving either extremity of the semi-ellipse.

6. Prove that the law of force in an equiangular spiral is that of the inverse cube of the distance; and explain why we cannot assert, conversely, that a body acted on by such a force will describe an equiangular spiral.

7. If the velocity at each point in a central orbit be equal to that in the equidistant circle, prove that the orbit is an equiangular spiral for an attractive force.

By Art. 89 the velocity in the equidistant circle $=\sqrt{Fr}$. Again, by Art. 158, the velocity in the orbit $=\sqrt{F\gamma}$; therefore $r=\gamma=p\frac{dr}{dp}$. Hence $\frac{dr}{r}=\frac{dp}{p}$; therefore $r=kp$, and consequently the orbit is an equiangular spiral.
If the force be repulsive, the orbit is an equilateral hyperbola.

8. In general, if the velocity at each point in a central orbit be in a constant ratio to that in an equidistant circle, find the law of force and the equation of the orbit.

Let the constant ratio be represented by $1:\sqrt{n}$; then, as in preceding example, we have

$$r=np\frac{dr}{dp};\ \text{hence}\ p=kr^n.$$

From this it is easily seen, as in Art. 158, that the equation of the orbit is of the form

$$r^{n-1}=a^{n-1}\cos(n-1)\theta.$$

The law of force is readily found; for, in general,

$$F\propto\frac{1}{p^2\gamma}\propto\frac{1}{p^2r}\propto\frac{1}{r^{2n+1}}.$$

9. In the same case show that the velocity at each point in the orbit is that due to motion from an infinite distance, subject to the central force.

Here $v^2 = \frac{Fr}{n} = \frac{\mu}{nr^{2n}}$; hence, by (17) Art. 157, the velocity is that due to an infinite distance.

10. When the velocity and direction of motion at any point, as well as the centre and intensity of the force, are given, show how to find the radius of curvature of the orbit at the point.

11. A body is acted on by two attractive centres of force, of equal intensity; and also by a repulsive force from another centre, of double the intensity; the forces varying directly as the distance. Prove that the orbit is a parabola, and show how to construct its focus and directrix when the initial velocity and direction of motion are given.

12. If a repulsive force vary as the inverse square of the distance, prove that the orbit is a branch of a hyperbola, having the centre of force in the focus external to the orbit.

13. A particle is acted on by a central repulsive force, which varies as the n^{th} power of the distance. If the velocity at any point be that due to motion from the centre of force; find the equation of the path.

Here, by (20), Art. 157, we have

$$v^2 = \frac{2\mu}{n+1} r^{n+1}, \text{ or } \frac{1}{p^2} = \frac{2\mu}{(n+1)h^2} r^{n+1};$$

or

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{k^2}{u^{n+1}}, \text{ where } k^2 = \frac{2\mu}{h^2(n+1)};$$

therefore

$$\frac{u^{\frac{n+1}{2}}\,du}{\sqrt{k^2 - u^{n+3}}} = d\theta;$$

hence we get

$$u^{\frac{n+3}{2}} = a \cos \frac{n+3}{2}\theta.$$

14. If the velocity at each point in a central orbit varies directly as the distance from the centre of force, prove that the orbit is an equilateral hyperbola, and find the law of force.

15. Show that the velocity at any point in a focal parabola is to that in the equidistant circle as $\sqrt{2} : 1$.

16. If the law of force be that of the inverse cube of distance, investigate the different varieties of orbit described.

Let $F = \mu u^3$, then equation (23) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}u; \text{ or } \frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu}{h^2}\right)u = 0.$$

Accordingly, the equation of the orbit depends on the sign of $1 - \frac{\mu}{h^2}$ and therefore on the initial circumstances of the motion.

Suppose the particle projected initially at the distance R, with the velocity V, and in a direction which makes the angle ω with R; then

$$h = VR \sin \omega.$$

Again, if V' be the velocity in the equidistant circle, we have, Art 89,

$$V'^2 = \frac{\mu}{R^2}; \quad \therefore \quad \frac{\mu}{h^2} = \frac{\mu}{V^2 R^2 \sin^2\omega} = \left(\frac{V'}{V \sin \omega}\right)^2.$$

Hence $1 - \frac{\mu}{h^2}$ is positive, zero, or negative, according as $V \sin \omega$ is greater, equal to, or less than V', the velocity in an equidistant circle.

(1) Let $V \sin \omega > V'$. In this case $1 - \frac{\mu}{h^2}$ is positive—equal k^2 suppose—and the equation may be written

$$\frac{d^2u}{d\theta^2} + k^2 u = 0.$$

The integral of this is of the form

$$u = A \cos (k\theta + \alpha).$$

A is plainly the maximum value of u; and therefore corresponds to an apsidal distance. Let a be this distance, and, if θ be measured from the apsidal line, the equation of the orbit is

$$r \cos k\theta = a. \tag{1}$$

(2) Let $V \sin \omega = V'$, then $1 - \frac{\mu}{h^2} = 0$, and we have

$$\frac{d^2u}{d\theta^2} = 0,$$

which gives $u = A(\theta + \alpha)$; and the equation of the orbit is reducible to

$$r\theta = \text{constant}, \tag{2}$$

which represents the hyperbolic spiral.

(3) Let $V \sin \omega < V'$. If we multiply the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} u,$$

by $2\,du$, and integrate, we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2} u^2 + c,$$

where c is constant.

Hence $$v^2 = \mu u^2 + h^2 c.$$

Substituting the initial values, this gives

$$c = \frac{V^2 - V'^2}{h^2};$$

therefore $$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2} u^2 + \frac{V^2 - V'^2}{h^2}. \tag{3}$$

The apse is determined by making $\frac{du}{d\theta} = 0$; consequently, since $\frac{\mu}{h^2} > 1$, the orbit has or has not an apse according as V is less or greater than V'.

Hence, if the *initial velocity be less than that in an equidistant circle*, the orbit is apsidal.

Suppose a to be the corresponding apsidal distance, then

$$\frac{V'^2 - V^2}{h^2} = \frac{1}{a^2}\left(\frac{\mu}{h^2} - 1\right),$$

and, making $\frac{\mu}{h^2} - 1 = k^2$, equation (3) becomes

$$\left(\frac{du}{d\theta}\right)^2 = \frac{k^2}{a^2}(a^2u^2 - 1);$$

therefore $$\frac{du}{d\theta} = \frac{k}{a}\sqrt{a^2u^2 - 1}; \quad \text{or} \quad \frac{a\,du}{\sqrt{a^2u^2 - 1}} = k\,d\theta.$$

The integral of this is

$$k\theta + \alpha = \log\left(au + \sqrt{a^2u^2 - 1}\right).$$

But if θ be measured from the apse, we have $\theta = 0$ when $au = 1$. Consequently $\alpha = 0$, and we have

$$au + \sqrt{a^2u^2 - 1} = e^{k\theta}.$$

Hence $$au = \tfrac{1}{2}\left(e^{k\theta} + e^{-k\theta}\right).$$

Next, if the initial velocity be equal to that in the equidistant circle, (3) becomes

$$\frac{du^2}{d\theta^2} = k^2u^2, \quad \text{or} \quad \frac{du}{d\theta} = ku;$$

hence $$u = ae^{k\theta},$$

the equiangular spiral (Ex. 7).

Again, if V be greater than V', let

$$\frac{V^2 - V'^2}{h^2} = k^2\beta^2,$$

and equation (3) becomes

$$\left(\frac{du}{d\theta}\right)^2 = k^2(u^2 + \beta^2), \quad \text{or} \quad \frac{du}{d\theta} = k\sqrt{u^2 + \beta^2}.$$

This, when integrated as above, gives

$$u + \sqrt{u^2 + \beta^2} = Ae^{k\theta},$$

and the curve is represented by the equation

$$2u = Ae^{k\theta} - \frac{\beta^2}{A}e^{-k\theta}.$$

The value of A can be readily determined from the initial conditions.

17. In elliptic motion about a centre of force in a focus, prove that $\int v\,ds$, taken through any arc, is proportional to the area subtended by the arc at the empty focus.

18. Prove that the expression for the central attraction for any law of force may be written in the form

$$F = \frac{h^2}{r^3} - \frac{d^2r}{dt^2}.$$

If we change the sign in the expression for the acceleration along the radius vector in Art. 29, we get

$$F = r\left(\frac{d\theta}{dt}\right)^2 - \frac{d^2r}{dt^2}.$$

This assumes the proposed form on substituting for $\frac{d\theta}{dt}$ its value $\frac{h}{r^2}$.

19. What would be the motion of a projectile if the force of gravity varied inversely as the cube of the height above a horizontal plane?

Here the path evidently lies in a vertical plane.

If the line of intersection of this plane with the horizontal plane be taken as the axis of x, and a vertical line as axis of y, the equations of motion may be written

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -\frac{\mu}{y^3};$$

therefore

$$\frac{dx}{dt} = c, \quad \left(\frac{dy}{dt}\right)^2 = \frac{\mu}{y^2} + c',$$

where c and c' are constants which depend on the initial circumstances of the motion. Consequently

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{c^2}\,\frac{\mu + c'y^2}{y^2},$$

or

$$\frac{y\,dy}{\sqrt{\mu + c'y^2}} = \frac{dx}{c}.$$

Hence we get

$$\sqrt{\mu + c'y^2} = c'\frac{x}{c} + \text{const.}$$

Consequently the path is an ellipse or a hyperbola according as the sign of c' is negative or positive. The path is a parabola if $c' = 0$.

20. Prove by Newtonian methods that, if two bodies attract one another according to any law, they describe similar figures about their centre of inertia and about one another.

Neglecting the obliquity of the ecliptic, and the inclination and the eccentricity of the lunar orbit, show that, if we take the Sun's distance as 390 times that of the Moon, the Earth's mass as 79 times that of the Moon, and the lunar synodic period as 30 mean solar days, then the solar day is, to a near approximation, shorter at full Moon than at new Moon by one 468,000th part of a mean solar day. *Camb. Trip.*, 1882.

21. A material particle, moving freely in a plane, being supposed to describe a conic under the action of a central force emanating from any point in the plane; show that the force varies directly as the distance from the point, and inversely as the cube of its distance from the polar of the point with respect to the curve.

22. In free motion in a plane under the action of a central force varying according to any law, state and prove the effect on the trajectory (and on the motion in it) of an additional force emanating from the same centre, and varying inversely as the cube of the distance.

23. An ellipse of eccentricity e and a parabola have a common focus and latus rectum; and equal particles describe them under the action of forces, to the common focus, of the same absolute intensity. If the particles moving in the same direction meet at one extremity of the common latus rectum and coalesce, prove that their subsequent path will be an ellipse of eccentricity $\frac{1}{2}(1 \pm e)$, according as both foci of the ellipse do or do not lie within the parabola; and find its major axis. What will the path be, if the particles be moving in opposite directions when they meet? *Camb. Trip.*, 1879.

24. A body is revolving in an ellipse whose eccentricity is $> \frac{1}{2}$, under the action of a force tending to the focus S, and when it is at a distance SP from S equal to the latus rectum, a blow is given to it perpendicular to SP, such that its new direction is perpendicular to the major axis. Show that the dimensions of the orbit are unaltered, but that the major axis is turned through an angle SPH, where H is the empty focus. *Id.* 1882.

25. Find the laws of attraction for which the trajectories described round a centre of force are closed orbits. (Bertrand, *Comptes rendus*, 1873.)

If $2\mu \int \frac{\phi(u)\,du}{u^2} = F(u) + \text{const.}$, Equation (23) gives

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{h^2} F(u) + c, \qquad (a)$$

where c is an arbitrary constant; therefore

$$\theta = \pm \int \frac{du}{\sqrt{\frac{1}{h^2} F(u) + c - u^2}}.$$

Again, let α, β represent the values of u which correspond to the apsidal distances; then α and β are roots of the equation

$$\frac{1}{h^2} F(u) + c - u^2 = 0.$$

Accordingly we must have

$$\frac{1}{h^2} F(\alpha) + c - \alpha^2 = 0, \quad \frac{1}{h^2} F(\beta) + c - \beta^2 = 0,$$

and if θ_0 be the apsidal angle, we get, abstraction being made of the sign,

$$\theta_0 = \int_\alpha^\beta \frac{du}{\sqrt{\frac{1}{h^2} F(u) + c - u^2}}. \qquad (b)$$

Assuming $m\theta_0 = \pi$, then for a closed orbit m must be a commensurable number (Art. 180).

If $\frac{1}{h^2}$ and c be eliminated by aid of the two preceding equations, we obtain

$$\theta_0 = \int_\alpha^\beta \frac{du\sqrt{F(\beta) - F(\alpha)}}{\sqrt{\alpha^2 F(\beta) - \beta^2 F(\alpha) + (\beta^2 - \alpha^2) F(u) - u^2 (F(\beta) - F(\alpha))}} = \frac{\pi}{m}; \qquad (c)$$

an equation which should hold for all values of α and β.

To determine the form of the function F, we suppose α and β very nearly equal, in which case the orbit is approximately circular.

Hence, from Art. 183, we get

$$m = \sqrt{k} = \sqrt{1 - \frac{\alpha F''(\alpha)}{F'(\alpha)}}, \text{ or } \frac{\alpha F''(\alpha)}{F'(\alpha)} = 1 - m^2.$$

Let $F'(\alpha) = V$, then $\frac{dV}{V} = (1 - m^2)\frac{d\alpha}{\alpha}$; hence

$$F'(\alpha) = V = C\alpha^{1-m^2},$$

where C is an arbitrary constant. (d)

From this we get $$F(\alpha) = \frac{C}{2 - m^2}\alpha^{2-m^2} + \text{const.}$$

We may assume the latter constant to be zero, since it disappears when we substitute in equation (c).

Again, since $$2\mu\frac{\phi(u)}{\quad} = F'(u), \quad \phi(u) = C_1 u^{3-m^2},$$

where C_1 is arbitrary.

We next proceed to determine m, from the condition that (c) must be satisfied for all values of α and β.

(1) Let $m^2 < 2$, and make $\alpha = 0$, and $\beta = 1$, then

$$F(\alpha) = 0, \quad F(\beta) = \frac{C}{2 - m^2}.$$

Substituting in (c), we obtain

$$\int_0^1 \frac{du}{\sqrt{u^{2-m^2} - u^2}} = \frac{\pi}{m}.$$

Again, if $u^{m^2} = z$,

$$\int_0^1 \frac{du}{u\sqrt{u^{-m^2} - 1}} = \frac{1}{m^2}\int_0^1 \frac{dz}{\sqrt{z(1-z)}} = \frac{\pi}{m^2}$$

The condition gives $\frac{\pi}{m} = \frac{\pi}{m^2}$; therefore $m = 1$. Accordingly, the force varies as u^2, or as $\frac{1}{r^2}$.

(2). Let $m^2 > 2$; then, if $\alpha = 0$, we have $F(\alpha) = F(0) = -\infty$; and if $\beta = 1$, we have $F(\beta) = F(1) = \dfrac{C}{2 - m^2}$. Substitute in ($c$), and divide the numerator and denominator by $\sqrt{F(\beta) - F(\alpha)}$, and it becomes

$$\int_0^1 \frac{du}{\sqrt{1 - u^2}} = \frac{\pi}{m}; \quad \text{or } \frac{\pi}{2} = \frac{\pi}{m}; \quad \therefore m = 2,$$

n which case the force varies directly as the distance.

Hence, as M. Bertrand observes, "parmi les lois d'attraction qui supposent l'action nulle à une distance infinie, celle de la nature est la seule pour laquelle un mobile lancé *arbitrairement*, avec une vitesse inférieure à une certaine limite, et attiré vers un centre fixe, décrive néccssairement autour de ce centre une courbe fermée. Toutes les lois d'attraction *permettent* des orbites fermées, mais la loi de la nature est la seule qui les *impose*."

CHAPTER VIII.

CONSTRAINED MOTION—MOTION IN A RESISTING MEDIUM.

SECTION I.—*Constrained Motion.*

185. **Movement on a Fixed Curve.**—When a particle is constrained to move, without friction, on a given fixed curve, the problem reduces to the determination of the velocity at any instant, as well as of the normal reaction of the curve. The motion may in this case be regarded as free by the introduction of the force of reaction of the curve, in addition to the external forces.

Hence, if N represents the normal reaction, the general equations of motion may be written, when referred to a rectangular system of axes,

$$m\frac{d^2x}{dt^2} = X + N\cos\alpha,\ m\frac{d^2y}{dt^2} = Y + N\cos\beta,\ m\frac{d^2z}{dt^2} = Z + N\cos\gamma, \tag{1}$$

where α, β, γ are the angles the normal reaction makes with the axes of coordinates; and X, Y, Z are the components of the external force, parallel to the axes of coordinates, respectively. If the first equation be multiplied by dx, the second by dy, and the third by dz, we get, on addition,

$$m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) = X\,dx + Y\,dy + Z\,dz, \tag{2}$$

since $\cos\alpha\,dx + \cos\beta\,dy + \cos\gamma\,dz = 0$, as the direction of N is perpendicular to the tangent to the curve.

This gives on integration

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \int(X\,dx + Y\,dy + Z\,dz) + \text{const.} \tag{3}$$

Hence the velocity is given by the same equation as in the case of unconstrained motion (Art. 128).

For a conservative system of forces (Art. 122), the velocity v at any point can generally be found from this equation. For, let $Xdx + Ydy + Zdz$ be the exact differential of the function $\phi(x, y, z)$; then if v' be the velocity at the point x', y', z', we have

$$\tfrac{1}{2}m(v^2 - v'^2) = \phi(x, y, z) - \phi(x', y', z'). \tag{4}$$

Hence the velocity at any point is independent of the path described; and, accordingly, if different curves be drawn joining any two points, a particle starting from one of these points with a given velocity would arrive at the other point with the same velocity whatever path it described.

Two of the preceding equations (1) are sufficient for a plane curve; for in this case N acts in the plane of the curve, and, by taking the axes of x and y in that plane, the third equation will disappear.

In the case of a central force, represented by $\mu\phi'(r)$, we have, as in Art. 128,

$$\tfrac{1}{2}m(v^2 - v'^2) = \mu(\phi(r) - \phi(r')). \tag{5}$$

Again, as in Art. 113, it is readily seen that the pressure on the curve in any case is the resultant of the centrifugal force and the normal component of the external forces.

The particle will leave the curve at the point for which the normal reaction becomes zero.

EXAMPLES.

1. A particle is constrained to move in a circle under the influence of a repulsive force, acting from a point on the circumference, and varying as the distance: find the pressure on the curve, the initial position being at the centre of force, and the particle starting from a state of rest.

Ans. $\dfrac{3\mu r^2}{2a}$, where r is the distance from the centre of force, and a the radius of the circle.

2. A particle is constrained to move in a logarithmic spiral, and is attracted to the pole of the spiral by a force varying inversely as the square of the distance. If the particle start from rest at the distance a from the pole, find the time of describing any portion of the curve.

Let μ denote the absolute force, then, by (5), we have

$$v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right),$$

or $$\frac{ds}{dt} = \sqrt{2\mu}\sqrt{\frac{1}{r} - \frac{1}{a}}.$$

Again, if $r = ce^{\frac{\theta}{k}}$ be the equation of the spiral, we have

$$\frac{ds}{dt} = \frac{dr}{dt}\sqrt{1 + k^2};$$

therefore $$\frac{dr}{dt} = \sqrt{\frac{2\mu}{1 + k^2}}\sqrt{\frac{1}{r} - \frac{1}{a}}.$$

Integrating, as in Art. 140, we get for the time of motion from the distance a to the distance r,

$$t = \sqrt{\frac{a(1 + k^2)}{2\mu}}\left(a\cos^{-1}\sqrt{\frac{r}{a}} + \sqrt{r(a - r)}\right).$$

Also the whole time of motion to the centre is $\frac{\pi}{2}\sqrt{\frac{a^3(1 + k^2)}{2\mu}}$.

It is readily seen that the problem of constrained motion in a logarithmic spiral, under the action of any central force directed to its pole, is reducible to free rectilinear motion under the action of a corresponding central force in the line of motion.

3. A particle under the action of gravity moves down the inner side of a smooth ellipse whose axis major is vertical. Being given its initial velocity, find where it will leave the ellipse.

Taking the centre as origin, and the axis major as axis of x, the value of x at the required point is given by the equation

$$2d = 3x - x^3\frac{e^2}{a^2},$$

where d is the height of the level line to which the velocity at each point is due above the centre.

4. In the same question find the least velocity at the lowest point of the ellipse in order that the particle should make a complete revolution in the curve.

Ans. $\sqrt{ga(5 - e^2)}$.

186. **Theorem of M. Ossian Bonnet.**—If masses m, m', m'', &c., respectively subject to the action of forces, F, F', F'', &c., and starting all in the same direction from a point A, with velocities v_0, v_0', v_0'', &c., describe the same curve

ACB; then the same path will also be described by the mass M, when projected from the same point in the same direction, and subject to the action of all the forces, F, F', F'', &c., provided the initial *vis viva* MV_0^2 is equal to

$$mv_0^2 + m'v'^2_0 + m''v''^2_0 + \&c.,$$

the sum of the *vires vivæ* of the different masses. (Bonnet, *Liouville's Journal*, 1844.)

For, suppose the particle M constrained to move in the curve ACB, and let N be the normal reaction at any point; then, if the components of F, parallel to a rectangular system of axes, be represented by X, Y, Z, respectively, those of F', by X', Y', Z', &c.; then, from (1), we have

$$M\frac{d^2x}{dt^2} = X + X' + X'' + \&c. + N\cos\alpha,$$

$$M\frac{d^2y}{dt^2} = Y + Y' + Y'' + \&c. + N\cos\beta,$$

$$M\frac{d^2z}{dt^2} = Z + Z' + Z'' + \&c. + N\cos\gamma,$$

and, as in (2), we have

$$d(MV^2) = 2dx\,\Sigma X + 2dy\,\Sigma Y + 2dz\,\Sigma Z.$$

But if v, v', v'', &c., be the respective velocities of m, m', m'', &c., at the same point, in the partial movements, we have

$$d(mv^2) = 2(Xdx + Ydy + Zdz),$$

$$\&c., \qquad \&c., \qquad \&c.$$

Hence $\quad d(MV^2) = d(mv^2 + m'v'^2 + m''v''^2 + \&c.)$;

therefore $\quad MV^2 = \Sigma(mv^2) + \text{constant},$

or $MV^2 = \Sigma mv^2$, from our hypothesis.

It is now easy to prove that the normal pressure N is zero at each point, and consequently that M would describe the curve ACB freely, under the combined action of all the forces.

For the force N is equal and opposite to the resultant of the centrifugal force, $\frac{MV^2}{\rho}$, and the several normal components of the forces, F, F', F'', &c. Again,

$$\frac{MV^2}{\rho} = \frac{mv^2}{\rho} + \frac{m'v'^2}{\rho} + \frac{m''v''^2}{\rho} + \text{\&c.};$$

but $\frac{mv^2}{\rho}$, $\frac{m'v'^2}{\rho}$, &c., are respectively equal and opposite to the normal components of F, F', F'', &c., respectively, because m, m', &c., describe the path ACB freely.

Hence there is equilibrium between the centrifugal force $\frac{MV^2}{\rho}$ and the total normal component of F, F', F'', &c.; and consequently $N = 0$.

In general, if the initial velocity of M do not satisfy the equation $MV_0^2 = \Sigma mv_0^2$, the normal pressure on the path ACB will vary directly as the curvature. For, from the preceding analysis,

$$N = \frac{MV^2 - \Sigma mv^2}{\rho} = \frac{MV_0^2 - \Sigma mv_0^2}{\rho}. \tag{6}$$

Also, if one of the forces (F' suppose) be changed into its opposite, it is readily seen that the preceding theorem still holds, provided we change the sign of the corresponding term ($m'v'^2$) in the expression $\Sigma(mv^2)$.

EXAMPLES.

1. A particle constrained to move in an ellipse is acted on by an attractive force directed to one focus, and a repulsive force from the other, whose intensities vary as the inverse square of the distance: if the absolute intensities of th forces be equal, find the pressure on the ellipse at any point during the motion.

2. Hence show that a particle placed at equal distances from two such centres of force will describe a semi-ellipse, under their joint action.

3. A particle moves under the attraction of two forces directed to the fixed points A and B, each varying according to the law of nature, and a third force, varying directly as the distance, directed to C the middle point of AB; show that the particle can be projected from any point so as to describe an ellipse having A and B as its foci. Lagrange, *Méc. Anal.*, T. 2, § 83.

Ans. The initial velocity v_0 is given by the equation

$$v_0^2 = \frac{\mu f'}{af} + \frac{\mu' f}{af'} + \mu'' ff',$$

where μ, μ', μ'' denote the absolute forces for the centres A, B, C, respectively; f, f' the initial distances from A and B; and a the semiaxis major of the ellipse. The initial direction of motion must bisect the external angle formed by the lines joining A and B to the point of projection.

4. In the same case, if the particle be constrained to move in the ellipse, find the reaction R at any point during the motion.

$$\textit{Ans.}\ R\rho = m\left(\frac{\mu f'}{af} + \frac{\mu' f}{af'} + \mu'' ff' - v_0^2\right),$$

where ρ is the radius of curvature at the point.

187. **Motion on a Fixed Surface.**—If a particle be constrained to move on a smooth surface, the general equations of motion are plainly, as in (1),

$$m\frac{d^2x}{dt^2} = X + N\cos\alpha,\quad m\frac{d^2y}{dt^2} = Y + N\cos\beta,\quad m\frac{d^2z}{dt^2} = Z + N\cos\gamma,$$

where α, β, γ are the direction angles of the normal to the surface.

It is obvious that in this case also the velocity at any point is determined by the equation

$$\tfrac{1}{2}mv^2 = \int (Xdx + Ydy + Zdz) + \text{const.} \qquad (7)$$

If gravity be the sole acting force, and the axis of z be taken in the vertical direction, our equations may be written

$$\frac{d^2x}{dt^2} = N\cos\alpha,\quad \frac{d^2y}{dt^2} = N\cos\beta,\quad \frac{d^2z}{dt^2} = N\cos\gamma - g. \qquad (8)$$

When the surface is one of revolution round a vertical axis, the normal at each point intersects that axis; and if n denote its length, we have

$$\cos\alpha = \frac{x}{n},\qquad \cos\beta = \frac{y}{n}.$$

Hence the two former equations give,

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0,$$

or, on integration,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c,$$

where c is a constant.

This equation shows that the point of projection on a horizontal plane describes equal areas in equal times round the point in which the axis of revolution meets the plane.

188. **Motion on a Spherical Surface.**—We shall apply what precedes to the motion of a particle under the action of gravity on a smooth sphere. This contains the general question of the motion of a simple pendulum, and is called the problem of the *spherical pendulum.* Taking the centre as origin, and the positive direction of the axis of z downwards, the equation of the sphere is

$$x^2 + y^2 + z^2 = a^2,$$

where a is the radius.

Also the general equations of motion may be written

$$\ddot{x} = N\frac{x}{a}, \quad \ddot{y} = N\frac{y}{a}, \quad \ddot{z} = N\frac{z}{a} + g,$$

adopting Newton's notation (Art. 23).

From the two first equations we get, as before,

$$x\dot{y} - y\dot{x} = c. \qquad (9)$$

Also, as in (7),

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = V_0^2 + 2g(z - a),$$

where V_0 represents the velocity corresponding to $z = a$.

Again, differentiating the equation of the sphere,

$$x\dot{x} + y\dot{y} + z\dot{z} = 0, \quad \text{or} \quad x\dot{x} + y\dot{y} = -z\dot{z}.$$

If this be squared and added to (9), when also squared, we get

$$(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) = c^2 + z^2\dot{z}^2.$$

Hence $$(a^2 - z^2)(V_0^2 + 2g(z - \alpha) - \dot{z}^2) = c^2 + z^2\dot{z}^2,$$

or $$a^2\dot{z}^2 = (a^2 - z^2)\{V_0^2 + 2g(z - \alpha)\} - c^2. \qquad (10)$$

The subsequent investigation is simplified by supposing V_0 to correspond to the *lowest* point in the path of the particle; for, since the motion at that point is horizontal, we have $\dot{z} = 0$ when $z = \alpha$, and consequently

$$c^2 = (a^2 - \alpha^2)V_0^2 = 2gh(a^2 - \alpha^2),$$

if h be the height to which the velocity V_0 is due.

Substituting this value for c^2 in (10), we get

$$a^2\dot{z}^2 = 2g(\alpha - z)\{z^2 + h(z + \alpha) - a^2\}.$$

Again, the expression $z^2 + h(z + \alpha) - a^2$ may be written $(z - \beta)(z + \gamma)$, where

$$h = \frac{a^2 - \beta^2}{\alpha + \beta}, \text{ and } \gamma = \frac{a^2 + \alpha\beta}{\alpha + \beta}. \qquad (11)$$

Accordingly

$$a^2\dot{z}^2 = 2g(\alpha - z)(z - \beta)(z + \gamma),$$

therefore $$a\dot{z} = a\frac{dz}{dt} = -\sqrt{2g(\alpha - z)(z - \beta)(z + \gamma)}. \qquad (12)$$

The negative sign must be taken since z diminishes with t, which is reckoned from the instant the particle is in its lowest position when z has its maximum value.

Also, when $z = \beta$ we have $\dot{z} = 0$, and the motion is again horizontal. It is readily seen that during the motion z must lie between the limits α and β; and consequently the path of the particle is a tortuous curve lying between two horizontal lesser circles on the sphere; we accordingly may assume

$$z = \alpha\cos^2\phi + \beta\sin^2\phi, \qquad (13)$$

and, substituting in (12), get

$$2a\frac{d\phi}{dt} = \sqrt{2g(\alpha\cos^2\phi + \beta\sin^2\phi + \gamma)}.$$

Hence, since $t = 0$ when $\phi = 0$, we have

$$t = a\sqrt{\frac{2(\alpha + \beta)}{g(a^2 + 2\alpha\beta + \alpha^2)}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}, \qquad (14)$$

where
$$k^2 = \frac{\alpha - \beta}{\alpha + \gamma} = \frac{\alpha^2 - \beta^2}{a^2 + 2\alpha\beta + \alpha^2}.$$

Consequently the time of motion depends on an elliptic function, and is reducible to that of the description of a corresponding arc in a vertical circle (Arts. 105, 112).

Again, if T denote the *period of a vibration*, that is, the time of motion from a lowest to a consecutive lowest position, we have

$$T = 2a\sqrt{\frac{2(\alpha + \beta)}{g(a^2 + 2\alpha\beta + \alpha^2)}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}.$$

It may be observed that when $\alpha = \beta$, we have $h = \dfrac{a^2 - \alpha^2}{2\alpha}$, and the question reduces to that of the conical pendulum, already considered in Art. 110.

Next let ψ be the angle that the vertical plane passing through the centre and the position of the particle at any instant makes with the plane of zx, then $y = x \tan\psi$; and consequently

$$c = x\frac{dy}{dt} - y\frac{dx}{dt} = x^2 \frac{d}{dt}\left(\frac{y}{x}\right) = x^2 \sec^2\psi \frac{d\psi}{dt}$$

$$= (x^2 + y^2)\frac{d\psi}{dt} = (a^2 - z^2)\frac{d\psi}{dt}. \qquad (15)$$

Also,
$$c = \sqrt{2gh(a^2 - \alpha^2)} = \sqrt{\frac{2g(a^2 - \alpha^2)(a^2 - \beta^2)}{\alpha + \beta}};$$

$$\therefore\ a\sqrt{\frac{(a^2 - \alpha^2)(a^2 - \beta^2)}{\alpha + \beta}} = -(a^2 - z^2)\sqrt{(\alpha - z)(z - \beta)(z + \gamma)}\frac{d\psi}{dz},$$

and the angle ψ is represented by an elliptic function of the third species, thus

$$\psi = a\sqrt{\frac{(a^2 - \alpha^2)(a^2 - \beta^2)}{\alpha + \beta}} \int_{\alpha}^{z} \frac{-dz}{(a^2 - z^2)\sqrt{(\alpha - z)(z - \beta)(z + \gamma)}}. \qquad (16)$$

In the projection of the path on the horizontal plane through the centre, the greatest and least distances from the centre correspond to the greatest and least values of z, i.e. to $z = \alpha$ and $z = \beta$. These are called the apsidal distances, and the corresponding angle, the *apsidal angle* of the path. If Ψ be the apsidal angle its value is represented by the integral

$$\Psi = a\sqrt{\frac{(a^2-\alpha^2)(a^2-\beta^2)}{\alpha+\beta}}\int_\beta^\alpha \frac{dz}{(a^2-z^2)\sqrt{(\alpha-z)(z-\beta)(z+\gamma)}}. \quad (17)$$

If the particle make a very small oscillatory motion round the lowest point, we may, as a first approximation, assume $k = 0$, in (14), and make $\alpha = a$, $\beta = a$; then we get

$$t = \sqrt{\frac{a}{g}}\,\phi.$$

Again, if $z = a\cos\theta$, $\alpha = a\cos\theta_0$, $\beta = a\cos\theta_1$, the equation

$$z = \alpha\cos^2\phi + \beta\sin^2\phi$$

becomes, neglecting powers of θ, θ_0, θ_1 beyond the second,

$$\theta^2 = \theta_0^2\cos^2\phi + \theta_1^2\sin^2\phi = \theta_0^2\cos^2 t\sqrt{\frac{g}{a}} + \theta_1^2\sin^2 t\sqrt{\frac{g}{a}}.$$

Also equation (15) gives, to the same degree of approximation $\left(\text{since } c = a^2\theta_0\theta_1\sqrt{\frac{g}{a}}\right)$,

$$\frac{d\psi}{dt}\sqrt{\frac{a}{g}} = \frac{\theta_0\theta_1}{\theta^2} = \frac{\theta_0\theta_1}{\theta_0^2\cos^2 t\sqrt{\frac{g}{a}} + \theta_1^2\sin^2 t\sqrt{\frac{g}{a}}}.$$

Hence, by integration,

$$\tan\psi = \frac{\theta_1}{\theta_0}\tan t\sqrt{\frac{g}{a}} = \frac{\theta_1}{\theta_0}\tan\phi.$$

Again, the polar equation of the projection is given by the equation

$$r^2 = a^2 - z^2 = a^2\theta^2$$

$$= a^2(\theta_0^2\cos^2\phi + \theta_1^2\sin^2\phi)$$

$$= a^2\frac{\theta_0^2 + \theta_1^2\tan^2\phi}{1 + \tan^2\phi} = \frac{a^2\theta_0^2(1 + \tan^2\psi)}{1 + \tan^2\phi}.$$

Hence $x = r\cos\psi = a\theta_0\cos\phi$; similarly $y = r\sin\psi = a\theta_1\sin\phi$. Consequently

$$\frac{x^2}{\theta_0^2} + \frac{y^2}{\theta_1^2} = a^2;$$

i. e. the horizontal projection of the path is, approximately, an ellipse, whose semiaxes are $a\theta_0$ and $a\theta_1$.

The next approximation is given in the following examples.

The general problem of the spherical pendulum appears to have been first fully discussed by Lagrange, see *Méc. Anal.*, t. 2, sect. 8.

EXAMPLES.

1. If the particle perform small oscillations about the lowest point, investigate its motion to an approximation of the second order.

It is here more convenient to transfer the origin to the lowest point on the sphere, and to take the positive direction of z *upwards*. Accordingly, we substitute $z = a - z'$, $\alpha = a - \alpha'$, $\beta = a - \beta'$, when equation (12) becomes

$$a\frac{dz'}{dt} = \sqrt{2g(z' - \alpha')(\beta' - z')\left(\frac{(2a - \alpha')(2a - \beta')}{2a - \alpha' - \beta'} - z'\right)}.$$

Hence, *removing the accents*, we get

$$t = \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz}{\sqrt{(z - \alpha)(\beta - z)\left(1 - \frac{z}{2a} + \frac{\alpha\beta}{2a(2a - \alpha - \beta)}\right)}},$$

where β and α represent the distances of the highest and lowest points in the path from the plane of xy.

Again, if α and β be both so small that their higher powers may be neglected, we obtain

$$t = \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz}{\sqrt{(z-\alpha)(\beta-z)\left(1-\dfrac{z}{2a}\right)}}$$

$$= \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz\left(1-\dfrac{z}{2a}\right)^{-\frac{1}{2}}}{(z-\alpha)^{\frac{1}{2}}(\beta-z)^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz}{\sqrt{(z-\alpha)(\beta-z)}} + \frac{1}{8\sqrt{ga}}\int_\alpha^z \frac{z\,dz}{\sqrt{(z-\alpha)(\beta-z)}},$$

neglecting the subsequent terms, since $\frac{z}{2a}$ is a very small fraction.

Hence, if $z = \alpha\cos^2\phi + \beta\sin^2\phi$, we get

$$t = \sqrt{\frac{a}{g}}\,\phi + \frac{1}{4\sqrt{ag}}\int_0^\phi (\alpha\cos^2\phi + \beta\sin^2\phi)\,d\phi$$

$$= \sqrt{\frac{a}{g}}\,\phi\left(1 + \frac{\alpha+\beta}{8a}\right) - \frac{\beta-\alpha}{8\sqrt{ag}}\sin\phi\cos\phi.$$

Consequently, if T be the whole time of motion from one lowest position to a consecutive one, we get

$$T = \pi\sqrt{\frac{a}{g}}\left(1 + \frac{\alpha+\beta}{8a}\right).$$

Again, to find the apsidal angle, to the same degree of approximation. Transforming the origin in equation (16) to the lowest position, we readily obtain

$$\psi = a\sqrt{\alpha\beta}\int_\alpha^z \frac{dz}{z(2a-z)\sqrt{(z-\alpha)(\beta-z)\left(1-\dfrac{z(2a-\alpha-\beta)}{(2a-\alpha)(2a-\beta)}\right)}}.$$

Hence, since as before we may take

$$\frac{2a-\alpha-\beta}{(2a-\alpha)(2a-\beta)} = \frac{1}{2a},$$

we get

$$\psi = a^{\frac{3}{2}}\sqrt{2\alpha\beta}\int_\alpha^z \frac{dz}{(2a-z)^{\frac{3}{2}}z\sqrt{(z-\alpha)(\beta-z)}}$$

$$= \tfrac{1}{2}\sqrt{\alpha\beta}\int_\alpha^z \frac{dz\left(1-\dfrac{z}{2a}\right)^{-\frac{3}{2}}}{z\sqrt{(z-\alpha)(\beta-z)}}$$

$$= \tfrac{1}{2}\sqrt{\alpha\beta}\int_\alpha^z \frac{dz}{z\sqrt{(z-\alpha)(\beta-z)}} + \frac{3}{8}\frac{\sqrt{\alpha\beta}}{a}\int_\alpha^z \frac{dz}{\sqrt{(z-\alpha)(\beta-z)}},$$

neglecting the subsequent terms, as before.

Substituting $\alpha\cos^2\phi + \beta\sin^2\phi$ for z, we obtain

$$\psi = \tan^{-1}\left(\sqrt{\frac{\beta}{\alpha}}\tan\phi\right) + \frac{3}{4}\frac{\sqrt{\alpha\beta}}{a}\phi.$$

Hence, taking ϕ between the limits 0 and $\frac{\pi}{2}$, the apsidal angle is given, approximately, by the equation

$$\Psi = \frac{\pi}{2}\left(1 + \frac{3}{4}\frac{\sqrt{\alpha\beta}}{a}\right).$$

This shows that in the approximate elliptic path the apse continually progresses.

Again, if p, q denote the small apsidal distances, or the semidiameters of the approximate elliptic path, we get

$$\Psi = \frac{\pi}{2}\left(1 + \frac{3}{8}\frac{pq}{a^2}\right).$$

Accordingly, the rate of progression of the apse varies approximately as the area of the projection of the path.

2. Prove that the pressure on the sphere is given by the equation

$$N = \frac{mg}{a}\left\{3z - \frac{2}{\alpha + \beta}(\alpha^2 + \alpha\beta + \beta^2 - a^2)\right\}.$$

3. If a particle be projected with a given velocity along the horizontal great circle of a smooth hollow sphere, find at what point its vertical velocity will be greatest.

Ans. $z = \dfrac{\sqrt{h^2 + 3a^2} - h}{3}$, h being the height due to the velocity of projection.

4. A particle is projected horizontally along the interior surface of a fixed smooth hemisphere, the axis of which is vertical, and vertex downwards. Given the point of projection, determine the velocity so that the particle may ascend exactly to the rim of the hemisphere.

Ans. $a\sqrt{\dfrac{2g}{\alpha}}$.

5. If a particle move on the interior surface of a paraboloid of revolution, whose axis is vertical, prove that the velocity at the highest point in the path is that due to the height of the lowest point above the vertex of the paraboloid; and similarly for the velocity at the lowest point.

6. In the last question show that the pressure at any point P varies as the curvature of the meridian at that point; and that the resolved vertical pressure is to the weight of the particle as $SL \times SM : SP^2$, where L and M are the highest and lowest points of the path, and S the focus.

SECTION II.—*Rectilinear Motion in a Resisting Medium.*

189. If a heavy body of mass m be supposed to fall vertically in a resisting medium, the resistance is a function of the velocity of the body, and the equation of motion may be written in the form

$$m\frac{d^2x}{dt^2} = mg - \phi(v).$$

It is usual to assume, with Newton, that $\phi(v) = AS \,.\, v^2$, where A is a constant depending on the density of the medium, and S is the area of the greatest section of the body perpendicular to the direction of motion.

Again, writing

$$AS = \frac{mg}{k^2}, \qquad (1)$$

the differential equation of motion becomes

$$\frac{d^2x}{dt^2} = g\left(1 - \frac{v^2}{k^2}\right), \text{ or } \frac{dv}{dt} = g\left(1 - \frac{v^2}{k^2}\right). \qquad (2)$$

It is obvious that the velocity increases so long as it is less than k: this gives k as the limit to which the velocity approaches, without, however, ever attaining to it, as the following analysis shows. For this reason k is called the *terminal velocity* of the body:—

Since

$$\frac{1}{k^2 - v^2} = \frac{1}{2k}\left\{\frac{1}{k+v} + \frac{1}{k-v}\right\},$$

the preceding equation gives

$$t = \frac{k}{2g}\log\left(\frac{k+v}{k-v}\right). \qquad (3)$$

No constant is added since we suppose t reckoned from the position of rest.

Also, as $\frac{dv}{dt} = v\frac{dv}{dx}$, we have $\frac{v\,dv}{k^2 - v^2} = \frac{g}{k^2}\,dx$;

therefore $$g\frac{x}{k^2} = C - \tfrac{1}{2}\log(k^2 - v^2) = \tfrac{1}{2}\log\left(\frac{k^2}{k^2 - v^2}\right), \qquad (4)$$

where x is measured from the position of rest.

To express x in terms of t, we get from (3),

$$\frac{k + v}{k - v} = e^{\frac{2gt}{k}};$$

hence $$\frac{k + v}{k} = \frac{2e^{\frac{2gt}{k}}}{e^{\frac{2gt}{k}} + 1}, \text{ and } \frac{k - v}{k} = \frac{2}{e^{\frac{2gt}{k}} + 1};$$

therefore $$\frac{k^2 - v^2}{k^2} = \frac{4}{\left(e^{\frac{gt}{k}} + e^{-\frac{gt}{k}}\right)^2},$$

and consequently

$$x = \frac{k^2}{g}\log\left(\frac{e^{\frac{gt}{k}} + e^{-\frac{gt}{k}}}{2}\right). \qquad (5)$$

This may also be written in the form

$$x = \frac{k^2}{g}\log\cosh\frac{gt}{k}.$$

In like manner from (3) we get the equation

$$v = k\tanh\frac{gt}{k}.$$

Also from (1) we get

$$k = \sqrt{\frac{W}{AS}},$$

where W denotes the weight of the body.

This shows that, W remaining the same, the value of k can be increased by diminishing the area of the transverse section. In the case of a homogeneous sphere, of radius r, we have

$W = \frac{4}{3}\pi r^3 p$, where p is the weight of a unit of volume; also $S = \pi r^2$; therefore

$$k = \sqrt{\frac{4pr}{3A}}.$$

Hence we can readily compare the motions of different spheres in the same resisting medium.

Next, for a body projected vertically upwards in a resisting medium, the equation of motion is

$$\frac{dv}{dt} = -g\left(1 + \frac{v^2}{k^2}\right), \tag{6}$$

whence
$$dt = -\frac{k^2}{g}\frac{dv}{v^2 + k^2}.$$

Accordingly, if V be the initial velocity, we find

$$t = \frac{k}{g}\left\{\tan^{-1}\frac{V}{k} - \tan^{-1}\frac{v}{k}\right\}.$$

From this equation the velocity at any instant can be determined.

Also, since $v = 0$ at the highest point, the time of ascent to that point is represented by $\frac{k}{g}\tan^{-1}\frac{V}{k}$.

Again
$$dx = -\frac{k^2}{g}\frac{v\,dv}{v^2 + k^2}.$$

Hence, if x be measured upwards from the point of projection, we have

$$x = \frac{k^2}{2g}\log\frac{V^2 + k^2}{v^2 + k^2}.$$

If h be the height of ascent, we get

$$h = \frac{k^2}{2g}\log\left(\frac{V^2 + k^2}{k^2}\right).$$

If the time t be reckoned from the highest point, we have

$$v = k\tan\frac{gt}{k}.$$

The downward motion is given by the former investigation.

EXAMPLES.

1. Find a vertical curve such that the time of describing any arc, measured from a fixed point, shall be equal to that of describing the chord of the arc.

Taking the origin at the fixed point, the time down a chord r, whose inclination to the vertical is θ, as in Art. 45, is

$$\sqrt{\frac{2r}{g\cos\theta}}.$$

Also the time of descending the arc is

$$\frac{1}{\sqrt{2g}}\int_{\theta_0}^{\theta}\frac{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}{\sqrt{r\cos\theta}}\,d\theta,$$

where θ_0 is the value of θ when $r = 0$.

Hence, since the times are the same for all chords, we get, by differentiation,

$$\frac{r\sin\theta+\cos\theta\frac{dr}{d\theta}}{\cos\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.$$

This gives $$\frac{1}{r}\frac{dr}{d\theta}=\cot 2\theta;$$

hence we get $$r^2 = a^2\sin 2\theta,$$

where a is a constant. Accordingly the curve is a Lemniscate.

2. Investigate the corresponding problem when the acting force is proportional to the distance from a fixed point.

Let A be the position of the fixed point, O the point of departure of the particle, P its position at any instant, $\theta = \angle POA$, $OA = a$; then we find, without difficulty, that the time t_1, of describing OP, when the absolute force is taken as unity, is given by

$$t_1=\sin^{-1}\frac{r-a\cos\theta}{a\cos\theta}+\frac{\pi}{2}.$$

Also the time of describing the arc OP is

$$t_2=\int\frac{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}{\sqrt{2ar\cos\theta-r^2}}\,d\theta.$$

Hence, since $t_1 = t_2$, we have

$$\sqrt{2ar\cos\theta-r^2}\,\frac{d}{d\theta}\left(\sin^{-1}\frac{r-a\cos\theta}{a\cos\theta}\right)=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2};$$

therefore $$\frac{dr}{d\theta} + r\tan\theta = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2};$$

from which we get $r^2 = a^2 \sin 2\theta$. This represents a lemniscate also, as in the previous question.

3. If the motion of a conical pendulum be slightly disturbed, prove that the period of a vibration is $\frac{2\pi a}{\sqrt{a^2+3b^2}}\sqrt{\frac{b}{g}}$, and the corresponding apsidal angle is $\pi\frac{a}{\sqrt{a^2+3b^2}}$, where b is the distance from the centre to the plane of the conical pendulum.

4. A particle is projected from a given point in a horizontal direction along the surface of a smooth sphere; find the velocity of projection in order that the particle should rise to a given height on the surface before commencing to descend.

5. A particle is constrained to move in a smooth circle, under the action of a central force which varies directly as the distance. If the time of describing any arc be constant, prove that its chord envelopes a circle.

Prof. Townsend, *Educ. Times*, 1875.

6. If a particle describe a curve freely under the combined action of the forces, F, F', &c., where F, F', &c., act along r, r', &c., prove that the equation

$$\gamma\phi d\left(\frac{F}{\phi}\right) + \gamma'\phi' d\left(\frac{F'}{\phi'}\right) + \text{\&c.} = 0$$

must be satisfied at every point of the curve, where ϕ, ϕ', &c., denote the forces respectively co-directional with F, F', &c., under which *singly* the given curve would be described; and γ, γ', &c., are the corresponding semichords of the circle of curvature at the point.

Prof. Curtis, *Messenger of Mathematics*, 1880.

Here, it is easily seen by equation (22), Art. 158, that

$$v^2 = F\gamma + F'\gamma' + \text{\&c.}$$

Also, by (13), Art. 157, $v\,dv = -F dr - F' dr' - \text{\&c.}$

Hence $$\Sigma(F d\gamma + \gamma dF) + 2\Sigma F dr = 0,$$

or $$\Sigma\{F(d\gamma + 2dr) + \gamma dF\} = 0.$$

Hence, in particular, we have

$$\phi(d\gamma + 2dr) + \gamma d\phi = 0, \text{ \&c.},$$

or $$\frac{d\gamma + 2dr}{\gamma} = \frac{-d\phi}{\phi}, \text{ \&c.};$$

$$\therefore \quad \Sigma\gamma\left(dF - \frac{F d\phi}{\phi}\right) = 0, \quad \text{or} \quad \Sigma\gamma\phi d\left(\frac{F}{\phi}\right) = 0.$$

This theorem plainly contains as a particular case that given in Art. 186.

7. Apply the preceding to the case of a conic described under the action of forces, F, F', directed to its foci.

Here $$\phi = \frac{\mu}{r^2}, \quad \phi' = \frac{\mu'}{r'^2}, \quad \gamma = \gamma';$$

therefore $$\frac{1}{r^2}\frac{d}{dr}(Fr^2)\,dr + \frac{1}{r'^2}\frac{d}{dr'}(F'r'^2)\,dr' = 0,$$

or, since $dr + dr' = 0$,

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = \frac{1}{r'^2}\frac{d}{dr'}(F'r'^2).$$

This is satisfied by the equations

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = f_1(r) + f_2(2a - r),$$

$$\frac{1}{r'^2}\frac{d}{dr'}(F'r'^2) = f_2(r') + f_1(2a - r'),$$

where f_1 and f_2 are both arbitrary functions.

If we assign the same form (f) to f_1 and f_2 we obtain as a particular solution

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = f(r) + f(2a - r),$$

or $$F = \frac{1}{r^2}\int r^2 \{f(r) + f(2a - r)\}\,dr, \text{ \&c.}$$

If any particular form be assigned to f, a corresponding form of F, as also of F', will result.

8. As an example of the preceding show that a particle can be made to describe an ellipse freely under the action of forces,

$$\lambda r + \frac{\mu}{r^2}, \quad \lambda r' + \frac{\mu'}{r'^2}$$

directed to its foci.

The student is referred to Professor Curtis' Paper for additional applications.

9. A spherical particle moves within a smooth rectilinear tube, which revolves about one extremity with a uniform angular velocity in a horizontal plane; find the motion of the particle.

Let ω be the angular velocity of the tube, and r the distance of the particle, at any time t, from the fixed extremity of the tube; then, since the force acting on the particle is always perpendicular to r, we have (Art. 28),

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0, \quad \text{or} \quad \frac{d^2r}{dt^2} - \omega^2 r = 0.$$

Hence $r = ce^{\omega t} + c' e^{-\omega t}$. If $r = a$, and $\frac{dr}{dt} = b$ when $t = 0$, we get

$$2\omega r = (a\omega + b) e^{\omega t} + (a\omega - b) e^{-\omega t}.$$

10. Consider the same problem if the tube be supposed to revolve uniformly in a vertical plane.

Here, if the time be reckoned from the instant that the tube was horizontal, the equation of motion is

$$\frac{d^2 r}{dt^2} - \omega^2 r = - g \sin \omega t.$$

The integral of this is

$$r = Ce^{\omega t} + C' e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t,$$

and the constants can be determined from the initial conditions.

11. Two spheres of the same diameter, but of different weights, fall freely in air; find the ratio of the maximum velocities they will attain, stating clearly what assumptions you make. *Lond. Univ.*, 1881.

12. Explain what is meant by the terminal velocity of a body in a resisting medium.

If the resistance vary as the square of the velocity and the body move in a vertical line, prove that at the time t the depth x of the body below its highest position is given by

$$x = \frac{\omega^2}{g} \log \sec \frac{gt}{\omega},$$

when ascending, and by

$$x = \frac{\omega^2}{g} \log \cosh \frac{gt}{\omega},$$

when descending; ω denoting the terminal velocity in the medium.

Lond. Univ., 1883.

13. If a body be projected vertically upwards in a resisting medium with its terminal velocity for the medium; determine the height of its ascent, and the time of reaching the highest point.

CHAPTER IX.

MOTION OF A RIGID BODY PARALLEL TO A FIXED PLANE.

Section I.—*Kinematics.*

190. **Rigid Body, Determination of its Position.**—A body is said to be *rigid* when its constitution is such that the relative position of its points with respect to each other is unalterable.

The position of a point in space is usually determined by means of three rectangular coordinates, and depends therefore upon three variable quantities independent of each other. It is easy to see that the position of a rigid body is determined by six independent variables.

In fact, the position in space of a definite point A of the body is determined by three independent variables; two more are required to determine the plane in space in which a definite plane a of the body passing through A should lie; and finally, one more is necessary to fix in this plane a definite line of the body passing through A, and lying in the plane a. When the position in space of every point of the plane a is determined, it is obvious that the positions of all points of the rigid body are completely determined, since perpendiculars from them on the plane a are invariable in magnitude.

191. **Degrees of Freedom.**—As six independent quantities are required to determine the position of a rigid body, such a body, if unacted on by any restraint, is said to have six *degrees of freedom.*

It is plain from what has been said, that if the positions of three points of a rigid body not lying on the same straight line are fixed, the position of every point of the body is determined.

192. **Motion of Translation.**—When a body moves in such a way that the elements of the paths described by its

different points are equal and parallel straight lines, the motion is said to be one of *translation.*

The path described by any one point of a body during a motion of translation is, in general, a curve, and it appears from the above definition that the curves described by the different points during any such motion are equal and similar.

In a motion of translation, the line joining any two definite points of the body remains parallel to its initial position.

If A_1, B_1, A_2, B_2, &c., are the positions occupied by A, B during the motion, AB is equal and parallel to A_1B_1, A_2B_2, and so on.

As the distances traversed by each point of the body are the same both in magnitude and direction, we may speak of *the motion of translation of the body*, and may compound any number of elementary motions in the same manner as for a point.

193. **Motion of Rotation.**—As already stated in Art. 93, when a body is moving in such a manner that each point is describing the arc of a circle having its centre on a fixed straight line to which its plane is perpendicular, the motion is said to be a *rotation*, and the fixed straight line passing through the centres of all the circles is called *the axis of rotation.*

In a motion of this kind every point of the body lying on the axis of rotation remains fixed during the motion.

All the lines in the body perpendicular to the axis of rotation turn through the same angle, which is called *the angular rotation*, or simply *the rotation* of the body.

Any definite line AB of the body which lies in a plane at right angles to the axis of rotation makes at the end of the motion an angle with its initial position, which is equal to the angular rotation of the body.

This readily appears as follows:—Join A to the point O in which the axis meets the plane in which AB lies; then, A' and B' being the new positions taken by A and B, since OA makes the same angle with AB which OA' makes with $A'B'$, the quadrilateral formed by OA, OA', AB and $A'B'$ can be inscribed in a circle, and therefore the angle between $A'B'$ and AB is equal to that between OA' and OA.

It is easy to see, that if two positions of a body have a straight line in common, the body can be moved from one of these positions to the other by a rotation round this line. For a definite line OA of the body, passing through a point O on the fixed line, and at right angles thereto in the first position, must remain at right angles to it in the second position OA'. Hence OA can be moved into the position OA' by a rotation round the fixed line; and as two straight lines in the body have then the situations belonging to the second position, the whole body must be in the second position.

194. **Motion Parallel to a Fixed Plane.**—When the paths described by the several points of a body during any motion are made up of elements, each of which is parallel to the same fixed plane, the motion of the body is said to be parallel to this plane.

If we consider any definite plane section of the body, which at the beginning of the motion is parallel to the fixed plane, this section must continue in the same plane throughout the motion, and its position at each instant determines the position of every point of the body. In order, therefore, to study the motion of a body moving parallel to a fixed plane, we have merely to investigate the motion of a plane figure in its own plane.

195. **Motion of a Plane Figure in its own Plane.**—A plane figure can be moved from any one position in its own plane to any other by giving it first a motion of translation, whereby any arbitrary point A of the figure is moved from its first position A_1 in space to its second position A_2, and then a motion of rotation round a perpendicular axis passing through A_2, whereby a definite line AB of the figure is moved into its final position in space A_2B_2. As the point A is perfectly arbitrary, the motion may be effected in an infinite variety of ways. The motion of translation to be given to the figure differs in general according as different points of the figure are chosen for A, but *the motion of rotation is in all cases the same.*

This readily appears from Art. 193, as the initial and final positions of any definite line of the figure are given, and the angle between them is, in all cases, the rotation of the body.

The results arrived at above depend upon the fact that the position of a plane figure in its own plane is completely determined by the position of one definite straight line of the figure. Hence also it appears that by properly selecting the point A, the motion of translation may in general be dispensed with altogether, or, in other words (*Differential Calculus*, Art. 293)—

A plane figure can be moved from any one position into any other in its own plane by a rotation round a point fixed in the plane.

In fact, BC being the original position of any definite line of the figure, and $B'C'$ its new position; if we join BB', bisect it, and erect a perpendicular, and do the same with CC', these two perpendiculars will, in general, determine by their intersection the point, a rotation round which effects the given change of position. If BB' be parallel to CC' this construction fails. Two cases then arise, according as BB' is equal to CC' or not. In the latter case, the intersection of BC and $B'C'$ is the centre of rotation. In the former the motion is one of pure translation, and the point is at infinity.

As a particular case, it follows that—

Two rotations effected successively round two parallel axes bring a body into the same position as a single rotation round an axis parallel to the two former, the single rotation being equal in magnitude to the sum of the two to which it is equivalent.

We see also that—

A rotation round any given axis brings a body into the same position as an equal rotation round a parallel axis through any arbitrary point, together with a motion of translation.

Hence it appears that—

Equal and opposite rotations effected successively round two parallel axes A and B are equivalent to a single motion of translation.

For, a rotation round A is equivalent to an equal rotation round B, together with a translation; therefore equal and opposite rotations round A and B are equivalent to equal and opposite rotations round B, together with a translation; but equal and opposite rotations round B destroy each other; therefore, &c.

196. **Composition of Velocities.**—Hitherto we have been considering displacements having a finite magnitude. In regard to such displacements the order in which the several motions are effected is, generally, of importance, and in order to arrive at definite results it is necessary to specify whether the successive axes of rotation are fixed in space or in the body. In Kinetics, we are for the most part concerned not only with the initial and final positions of a body, but also with the mode in which it passes from the one position to the other. It becomes then necessary to consider the infinitely small motions through which the body assumes its successive positions. Displacements effected in the same element of time divided by that element then become velocities, and the composition and equivalence of infinitely small displacements is tantamount to the composition and equivalence of velocities.

If the displacements received by a body be infinitely small, it is indifferent in what order rotations are effected round different axes fixed in space.

For, the changes in the coordinates of a point of the body, due to a displacement, are functions of the initial values of those coordinates. In the present case these changes are infinitely small, and therefore the changes resulting from the first displacement produce alterations in those which would have resulted from the second had it been given first, which are infinitely small quantities of the second order.

For a similar reason it is indifferent whether the axes be fixed in space or axes fixed in the body, whose positions at the commencement of the infinitely small motion coincide with those of the axes fixed in space.

When the order of two displacements is indifferent they may be regarded as simultaneous, and if the resultant displacement be such as to move the body from one position into the next successive position, it is the actual motion of the body.

The theorems of the last Article, when applied to infinitely small motions, give the following results:—

(1). The motion of a body parallel to a fixed plane consists at any instant of a velocity of rotation ω round an axis, passing through any arbitrary point A of the body, at right angles to the plane, and a velocity of translation parallel to the plane.

(2). At each instant there is one point O, called the instantaneous centre of rotation, which is such that a velocity of rotation ω round an axis through it represents the whole motion of the body.

If r be the distance from O to A, and v the velocity of translation which the body must be considered to possess when the axis of rotation is regarded as passing through A, v is at right angles to r, and equal to $r\omega$. The point O is found by drawing perpendiculars to the simultaneous directions of motion of any two points of the body.

(3). Two coexisting velocities of rotation round parallel axes are equivalent to a single velocity of rotation, equal to their sum, round an axis parallel to the two former, and dividing the distance between them inversely as the component velocities.

(4). Two equal and opposite velocities of rotation whose common magnitude is ω, round parallel axes, are equivalent to a velocity of translation, perpendicular to the plane of the axes, whose magnitude is $a\omega$, where a is the distance between the axes.

We see from what is stated above that velocities of rotation round parallel axes are compounded like parallel forces.

In considering rotations round parallel axes it is necessary to take into account not only the magnitudes of the rotations, but also their algebraical signs, or directions. The axis of rotation is supposed to be drawn from the feet of the spectator to his head, so that in estimating rotations the axis points towards the spectator. If two rotations round parallel axes, viewed from the same side of the plane perpendicular to the axes, are both in the same direction, they have like algebraical signs. The positive direction of rotation is of course arbitrary; but in the application of Analytic Geometry to rotational displacements it is usual to regard rotations as positive which bring a point—from the axis of X positive to the axis of Y positive, from Y positive to Z positive, and from Z positive to X positive. It follows from this, that if the axes of X, Y, and Z be drawn in the usual manner, a rotation opposite in direction to that of the hands of a watch is to be regarded as positive (Art. 85). Such a rotation is termed counterclockwise.

197. **Analytical Treatment of the Motion of a Plane Figure in its own Plane.**—When a point moves in a circle whose centre is the origin, we may assume

$$x = r\cos\psi, \quad y = r\sin\psi,$$

whence $$\frac{dx}{dt} = -r\sin\psi\frac{d\psi}{dt}, \quad \frac{dy}{dt} = r\cos\psi\frac{d\psi}{dt},$$

and putting $\frac{d\psi}{dt} = \omega$, we have

$$\frac{dx}{dt} = -y\omega, \quad \frac{dy}{dt} = x\omega, \qquad (1)$$

for the rotation of the point x, y round the origin.

Now let x', y' be the coordinates of any definite point of a plane figure referred to axes fixed in space; x, y those of any point of the figure referred to the same axes; ξ, η those of the same point of the figure referred to axes fixed in the body and meeting at the point $x'y'$: moreover, let the axis of ξ make the angle ψ with the axis of x; then

$$x = x' + \xi\cos\psi - \eta\sin\psi, \quad y = y' + \xi\sin\psi + \eta\cos\psi,$$

whence

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx'}{dt} - (\xi\sin\psi + \eta\cos\psi)\frac{d\psi}{dt} \\ \frac{dy}{dt} &= \frac{dy'}{dt} + (\xi\cos\psi - \eta\sin\psi)\frac{d\psi}{dt} \end{aligned}\right\}. \qquad (2)$$

Or, putting $$\frac{d\psi}{dt} = \omega,$$

$$\frac{dx}{dt} = \frac{dx'}{dt} - (y - y')\omega, \quad \frac{dy}{dt} = \frac{dy'}{dt} + (x - x')\omega. \qquad (3)$$

These equations show that the velocity of the point x, y is made up of two parts—one a velocity of translation, the other a velocity of rotation, as in (1), round an axis through x', y'.

For any other definite point, x'', y'' of the figure we have, in like manner,

$$\frac{dx}{dt} = \frac{dx''}{dt} - (y - y'')\,\omega'', \quad \frac{dy}{dt} = \frac{dy''}{dt} + (x - x'')\,\omega''.$$

Equating these values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to the former, and comparing the results with the equations

$$\frac{dx'}{dt} = \frac{dx''}{dt} - (y' - y'')\,\omega'', \quad \frac{dy'}{dt} = \frac{dy''}{dt} + (x' - x'')\,\omega'',$$

we see that $\omega'' = \omega$, or the velocity of rotation to be attributed to the body is the same whatever be the point through which the axis of rotation is supposed to pass.

If we put $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ in equations (2), we get the coordinates of the instantaneous centre of rotation, referred to axes fixed in the body. In like manner, equations (3) give the coordinates of the same point referred to axes fixed in space. If we call the coordinates of the instantaneous centre ξ_0, η_0, x_0, y_0, we have

$$\left.\begin{aligned} \xi_0 &= \frac{1}{\omega}\left(\frac{dx'}{dt}\sin\psi - \frac{dy'}{dt}\cos\psi\right), \\ \eta_0 &= \frac{1}{\omega}\left(\frac{dx'}{dt}\cos\psi + \frac{dy'}{dt}\sin\psi\right), \end{aligned}\right\} \qquad (4)$$

$$x_0 = x' - \frac{1}{\omega}\frac{dy'}{dt}, \quad y_0 = y' + \frac{1}{\omega}\frac{dx'}{dt}. \qquad (5)$$

If $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, ω, and ψ are known functions of the time t, we can find from equations (4), by eliminating t, the path described in the body by the instantaneous centre.

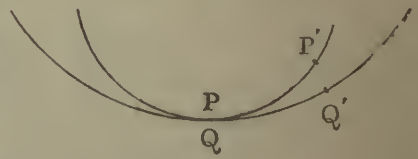
From equations (5) we can find in the same manner the path described by the instantaneous centre in space.

The former of these curves is called the *body centrode*, the latter the *space centrode*.

The student must carefully distinguish between the instantaneous centre and the point of the body which coincides with it at any instant. The latter has no velocity either in space or in the body; indeed its relative coordinates, ξ, η, are absolutely constant during the motion: the former (the instantaneous centre) has in general a velocity both in space and in the body.

198. **Pure Rolling.**—In pure rolling the points of one curve or surface come into contact successively with those of another, the relative tangential velocity of the points of contact being zero. If one curve or surface be fixed in space, the motion of the other consists of a series of rotations round axes through the successive points of contact (*Differential Calculus*, Art. 295).

For, let QQ' be the curve fixed in space, and PP' the one which rolls on it, P, P' being two consecutive points on the latter. By hypothesis, P has no velocity along the tangent at Q, and at the end of an infinitely short time P' coincides with Q', and the distance between P and Q is then an infinitely small quantity of the second order. Hence, while other points of the body have received infinitely small displacements of the first order, P has received one of the second order, and has, therefore, no velocity in any direction. Hence, during the instant under consideration the body must be rotating round an axis through P (Art. 193). It is obvious that the acceleration of P in the direction of the tangent at Q is zero; and it can be easily seen that its acceleration in the direction of the normal is in general finite, and equal to $U\omega$, where ω is the angular velocity of the body, and U is the velocity of the instantaneous centre of rotation, this point having moved during the instant from Q to Q' in space, and from P to P' in the body.



199. **Geometrical Representation of the Motion of a Body moving Parallel to a Fixed Plane.**—When a body is moving parallel to a fixed plane, if we can determine the space centrode and the body centrode, the motion of

the body is completely determined, as it consists of the rolling, without slipping, of the body centrode on the space centrode.

The geometrical applications of the principles laid down in the present and preceding Articles are numerous and important; but as they do not fall within the scope of the present treatise, the reader is referred for them to Chap. XIX. of the *Differential Calculus*, and to Minchin's *Uniplanar Kinematics*, Chap. III.

200. **Velocity of any Given Point of a Body.**—In Kinetics the motion of a body is usually regarded as made up of a motion of translation and a motion of rotation round an axis through the centre of inertia. It is sometimes important to determine the velocity of a given point of the body. In the case of motion parallel to a fixed plane this is readily done analytically by equations (3).

Otherwise, geometrically, if G be the centre of inertia through which the axis of rotation is regarded as passing, and the velocity of a definite point A of the body be required, we may apply round this point two equal and opposite rotations, whose common magnitude is ω. One of these, along with the equal and opposite rotation round the axis through G will produce a velocity of translation, and this, combined with the velocity of translation previously known, gives the velocity of A.

EXAMPLES.

1. Show directly that if a body have two equal and opposite velocities of rotation round two parallel axes, the motion of *any* point is at right angles to the plane containing the parallel axes, and the velocity of the point is equal to the distance between the axes multiplied by the angular velocity.

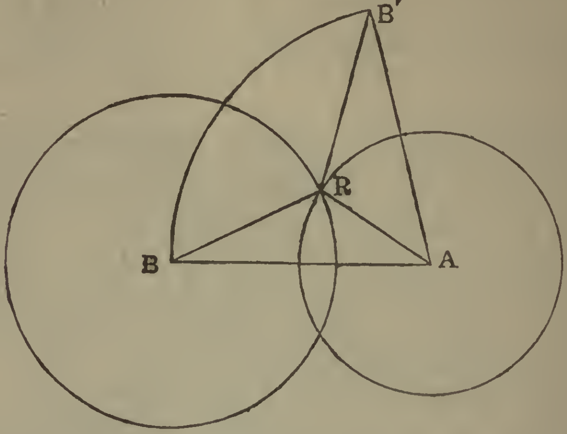
Draw a plane through the point at right angles to the two parallel axes. Describe circles round the axes passing through the point. The component velocities of the point are perpendicular and proportional to the radii of these circles, and the resultant velocity is, therefore, in the direction of the common chord, and proportional to the line joining the centres.

2. Prove that a velocity of rotation round any axis is equivalent to an equal velocity of rotation ω round a parallel axis, together with a velocity of translation ωa along a line at right angles to the plane containing the axes, the distance between which is a.

3. A body receives, in a given order, finite rotations round two parallel axes fixed in space. Determine the magnitude of the equivalent rotation, and the position of its axis.

4. If the parallel axes round which the body receives successive rotations be fixed not in space but in the body, determine the single rotation which would bring the body into the same position.

If A, B are the intersections of the axes fixed in space, with a plane at right angles, R that of the resultant axis, and α, β, ω, the magnitudes of the rotations round them, then $BAR = -\frac{1}{2}\alpha$, $ABR = \frac{1}{2}\beta$, and the resultant rotation $\omega = \alpha + \beta$, or, $\pm(\alpha - \beta)$, according as α and β are in the same or opposite directions. In the latter case its direction is the same as the greater of the two. If A and B' are the positions of the axes fixed in the body, $B'AR = \frac{1}{2}\alpha$, $AB'R = -\frac{1}{2}\beta$.



5. Two equal and opposite finite rotations round parallel axes bring a body into the same position as a single motion of translation. Determine the direction and magnitude of this translation.

The direction of translation is at right angles to a line which makes with AB or AB' an angle equal to $-\frac{1}{2}\alpha$, or $\frac{1}{2}\alpha$, and the magnitude of the translation $= 2AB \sin \frac{1}{2}\alpha$.

6. If the direction of the motion of each point of a body be always parallel to a fixed plane, the motion is equivalent to a succession of rotations round the generating lines of a cylinder fixed in space, which is at right angles to the fixed plane.

7. A plane area receives a motion of translation in its own plane whose components, parallel to the axes, are a and b; and a rotation θ round the point in the body which, at the beginning of the motion, coincides with the fixed origin. Determine the coordinates of the point a rotation round which would bring the body into the same position.

$$\textit{Ans.}\quad x = \frac{a \sin \frac{1}{2}\theta - b \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta}, \quad y = \frac{b \sin \frac{1}{2}\theta + a \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta}.$$

8. Show from these expressions that the amplitude of the rotation is the same as before.

If ϕ be the amplitude, $\quad \sin \frac{1}{2}\phi = \frac{1}{2}\,\dfrac{\sqrt{a^2 + b^2}}{\sqrt{x^2 + y^2}} = \sin \frac{1}{2}\theta.$

9. A plane area is moving in its own plane: determine the accelerations of any point in it parallel to the tangent and the normal drawn to the space centrode at the instantaneous centre of rotation.

Let x_0, y_0 be the coordinates of a point fixed in the body, ξ, η those of any point in the body referred to x_0, y_0 as origin, and to axes parallel to those of x, y; then

$$\frac{d\xi}{dt} = -\eta\omega, \quad \frac{d\eta}{dt} = \xi\omega,$$

ω being the angular velocity of the body; whence

$$\frac{dx}{dt} = \frac{dx_0}{dt} - \eta\omega,$$

$$\frac{dy}{dt} = \frac{dy_0}{dt} + \xi\omega,$$

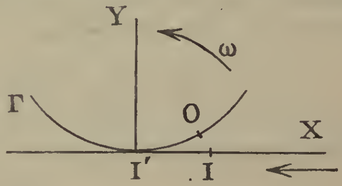
$$\frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} - \omega^2\xi - \frac{d\omega}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = \frac{d^2y_0}{dt^2} + \frac{d\omega}{dt}\xi - \omega^2\eta.$$

Call the centrode fixed in space C, that fixed in the body, Γ. The velocity of the point O of the body which coincides at any instant with the instantaneous centre of rotation is zero. At the next instant the instantaneous centre of rotation has moved to the consecutive position on each of the curves C and Γ. At the end of this instant O has a velocity in the normal to C equal to $\omega U dt$, where $U dt$ is the motion of the instantaneous centre along C. Hence the acceleration of O along the tangent to C is zero, and along the normal to C is $-\omega U$. Now, make x_0, y_0 coincide with O, and take as axes the tangent and normal to C, then

$$\frac{d^2x}{dt^2} = -\omega^2\xi - \frac{d\omega}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = -\omega U + \frac{d\omega}{dt}\xi - \omega^2\eta.$$



In the figure I, I' are consecutive positions of the instantaneous centre on the tangent to C, and the positive rotation is from X to Y.

10. Determine the points of the body which have at any instant (1) no acceleration parallel to the tangent to C; (2) no acceleration parallel to the normal at the instantaneous centre of rotation.

These points consist of two straight lines in the body at right angles to each other, the first of which passes through the instantaneous centre of rotation.

11. Determine at any instant the position of the point in the body having no acceleration.

It is the intersection of the two lines mentioned in the last example. If ρ, ρ' be the radii of curvature of C and Γ, if we put $\frac{1}{R} = \frac{1}{\rho} - \frac{1}{\rho'}$, and if α be the angle which the line of non-tangential acceleration (Ex. 10) makes with the axis of x, the coordinates of this point may be expressed in the form

$$\xi = R \sin\alpha \cos\alpha, \quad \eta = R \sin^2\alpha.$$

These expressions readily follow from the equations of Ex. 9, since it is easily seen that

$$\frac{U}{\omega} = R.$$

This point is called the acceleration-centre.

12. The acceleration of any point of the body is the same as if the body were turning round the acceleration-centre as an *absolutely fixed point.*

Transforming the origin to the acceleration-centre, we get

$$\frac{d^2x}{dt^2} = -\omega^2\xi - \frac{d\omega}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = +\frac{d\omega}{dt}\xi - \omega^2\eta;$$

but these are the accelerations which would result if the body were turning round the origin of $\xi\eta$ as an absolutely fixed point with the varying angular velocity ω from ξ to η.

13. All points of the body which have a common acceleration lie on a circle having the acceleration-centre as centre.

Square, and add the equations in the last example.

14. Find the points of the body for which the acceleration normal to the path described by the point is zero.

Take the centre of rotation as origin of $\xi\eta$, any point is describing a circle round it; hence the line joining the origin to $\xi\eta$ is the normal to the path of the latter; and if N be the normal acceleration, and r the distance from the instantaneous centre of rotation,

$$N = \frac{\xi}{r}\left(-\omega^2\xi - \frac{d\omega}{dt}\eta\right) + \frac{\eta}{r}\left(-\omega U + \frac{d\omega}{dt}\xi - \omega^2\eta\right) = -\omega^2\frac{\xi^2+\eta^2}{r} - \frac{\eta}{r}\omega U.$$

Hence at any instant the points for which $N = 0$ lie on a circle

$$\omega U\eta + \omega^2(\xi^2 + \eta^2) = 0,$$

which passes through the instantaneous centre of rotation, touches the curve C, and has a radius $= \frac{1}{2}R$. For the reason stated in Ex. 16 this circle is called the circle of inflexions.—*Differential Calculus*, Art. 290.

15. Determine the points of the system for which the acceleration along the path is zero.

They lie on a circle whose equation referred to the centre of rotation as origin is

$$\frac{d\omega}{dt}(\xi^2 + \eta^2) - \omega U\xi = 0,$$

which passes through the instantaneous centre of rotation and cuts the curve C orthogonally.

The theorems of the last two examples are due to Bresse (*Journal de l'école polytechnique*, t. xx.).

16. Determine at any instant the points of the body which are passing over points of inflexion on their respective paths.

They are the points having no normal acceleration (Ex. 14); for, as $\frac{v^2}{\rho}$ is then zero, and v not zero, ρ must be infinite.

17. Determine the coordinates of the acceleration centre referred—(1) to axes fixed in the body; (2) to axes fixed in space (*see* Article 197).

Let ξ_1, η_1, x_1, y_1 be the coordinates in question, then—

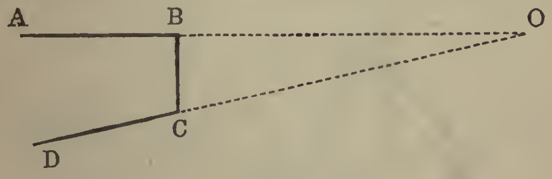
$$\left\{\left(\frac{d\omega}{dt}\right)^2+\omega^4\right\}\xi_1=\frac{d\omega}{dt}\left(\frac{d^2x'}{dt^2}\sin\psi-\frac{d^2y'}{dt^2}\cos\psi\right)+\omega^2\left(\frac{d^2x'}{dt^2}\cos\psi+\frac{d^2y'}{dt^2}\sin\psi\right),$$

$$\left\{\left(\frac{d\omega}{dt}\right)^2+\omega^4\right\}\eta_1=\frac{d\omega}{dt}\left(\frac{d^2x'}{dt^2}\cos\psi+\frac{d^2y'}{dt^2}\sin\psi\right)-\omega^2\left(\frac{d^2x'}{dt^2}\sin\psi-\frac{d^2y'}{dt^2}\cos\psi\right),$$

$$\left\{\left(\frac{d\omega}{dt}\right)^2+\omega^4\right\}(x_1-x')=-\frac{d\omega}{dt}\,\frac{d^2y'}{dt^2}+\omega^2\,\frac{d^2x'}{dt^2},$$

$$\left\{\left(\frac{d\omega}{dt}\right)^2+\omega^4\right\}(y_1-y')=\frac{d\omega}{dt}\,\frac{d^2x'}{dt^2}+\omega^2\,\frac{d^2y'}{dt^2}.$$

18. *ABCD* is a frame composed of three bars connected by joints at B and C. It is capable of moving in one plane, the points A and D being fixed. Determine the relation between the angular velocities of AB and DC.



At any instant B is moving in a circle round A, or at right angles to AB; and C, at right angles to DC. Hence the instantaneous centre of rotation of BC is O, the point of intersection of AB and CD; wherefore $AB\,.\,\omega_1=OB\,.\,\omega$, and $DC\,.\,\omega_2=OC\,.\,\omega$; hence $\frac{\omega_1}{\omega_2}=\frac{BO}{AB}\cdot\frac{DC}{CO}$ (Thomson and Tait).

19. A bar AB moves in one plane with given angular velocity round A, while at B it is freely jointed to another bar BC, whose extremity C is constrained to move along a fixed straight groove passing through A; find the velocity of C in any position.

Draw a perpendicular to AC at C, and let it meet AB in O; then O is the instantaneous centre of rotation of BC. If v be the velocity of C, and ω the angular velocity of AB, $v = AB \cdot \frac{OC}{OB} \cdot \omega = AP \cdot \omega$, where AP is drawn at right angles to AC to meet BC in P. For the further discussion of this question the reader is referred to Minchin, *Uniplanar Kinematics*, p. 47, or Goodeve, *Elements of Mechanism*, Chap. I. The arrangement of machinery mentioned in this example is called the crank and connecting rod.

20. A bar moves in a horizontal plane with uniform angular velocity round one extremity. To the other extremity a horizontal circle is attached. If the circle be regarded as rotating round its centre, what additional motion must it be considered to have?

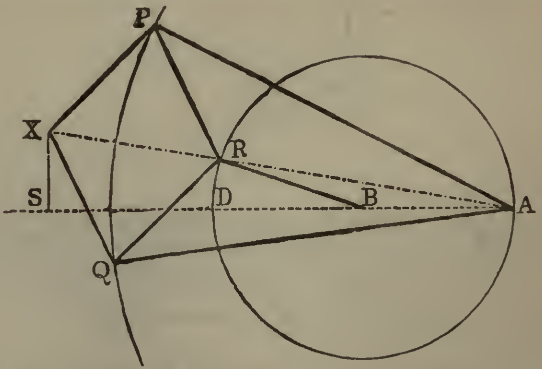
The circle has a velocity of translation at right angles to the bar, and equal to $a\omega$, where a is the distance of its centre from the fixed end of the bar; and a velocity of rotation ω round a vertical axis through the centre.

21. If two definite points of a plane figure are constrained to move along two straight lines in its plane which are fixed in space, the space centrode and the body centrode are circles, the former being double the latter (*Differential Calculus*, Art. 295).

22. In Peaucellier's arrangement find the relation between the velocity of the point describing the straight line and that of one of the adjacent corners of the parallelogram.

M. Peaucellier, in 1864, first succeeded in transforming circular into rectilinear motion by the following arrangement:—A and B are fixed points; AP and AQ are two equal bars which can turn freely round A; BR is another bar turning freely round B, and equal in length to AB; $QRPX$ is a jointed parallelogram composed of four equal bars turning freely round their points of intersection. If a motion be imparted to the system, the points P, Q, R describe circles. That the point X describes a straight line may be shown as follows:—

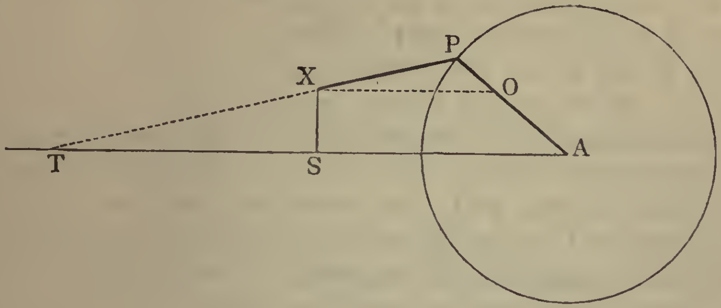
In any position of the system, since $\angle PRX = \angle QRX$, and $\angle PRA = \angle QRA$, XR and RA are in one straight line; then XPR being an isosceles triangle, and



PA a line drawn from the vertex to the base, $AR \cdot AX = AP^2 - RP^2 = \text{const.}$; wherefore X describes a curve which is the inverse, with respect to A as origin, of that described by R. Now the point R describes a circle which passes through A; hence X describes a straight line, perpendicular to AB at the point S, where

$$AS \,.\, AD = AP^2 - RP^2.$$

We proceed to find the relation between the velocities of P and X. Draw XO at right angles to SX; then O is the instantaneous centre of rotation of the bar PX.



Let $AP = a$, $PX = b$, BR (in former figure) $= c$; then ω being the angular velocity of AP, ω' the angular velocity of PX about O, and v the velocity of X; we have, since O is the instantaneous centre,

$$v = OX \,.\, \omega', \text{ and } OP \,.\, \omega' = AP \,.\, \omega;$$

therefore

$$v = \frac{OX}{OP} \,.\, AP \,.\, \omega = AT \,.\, \omega.$$

Again, if

$$PAT = \theta,\ PTA = \phi. \quad AT = a \sin\theta\,(\cot\theta + \cot\phi);$$

therefore

$$v = a \sin\theta\,(\cot\theta + \cot\phi)\,\omega,$$

where ϕ is given by the equation

$$a\cos\theta + b\cos\phi = \frac{a^2 - b^2}{2c}.$$

SECTION II.—*Kinetics of a Rigid Body, General Equations.*

201. **D'Alembert's Principle.**—If a system of material points connected together in any way, and subject to any constraints, be in motion under the influence of any forces, each point of the system has at any instant a certain acceleration. If now to each point an acceleration were applied equal and opposite to the actual acceleration of the point, the velocities of all the points of the system would become constant—in other words, each point would move as if free and unacted on by any force whatever—that is, the applied accelerations, the external forces, and the constraints and mutual or internal forces of the system, would equilibrate each other.

Stated in algebraical language, the principle which is given above may be enunciated as follows:—If the coordinates of any point of a material system be x, y, z, the mass there concentrated m, and the external forces there applied X, Y, Z; the system of forces,

$$X - m\frac{d^2x}{dt^2},\quad Y - m\frac{d^2y}{dt^2},\quad Z - m\frac{d^2z}{dt^2},$$

$$X' - m'\frac{d^2x'}{dt^2},\quad Y' - m'\frac{d^2y'}{dt^2},\quad Z' - m'\frac{d^2z'}{dt^2},\ \&c.,$$

acting at the points xyz, $x'y'z'$, &c., will be in equilibrium, in virtue of the constraints and mutual reactions of the system.

The force whose components are $-m\frac{d^2x}{dt^2}$, $-m\frac{d^2y}{dt^2}$, $-m\frac{d^2z}{dt^2}$ is called the force of inertia of the mass m, and D'Alembert's Principle (as stated in Article 69) simply expresses that—

The applied forces and the forces of inertia in any system are in equilibrium.

In applying D'Alembert's Principle we may, as in Statics, consider the constraints of the system either as geometrical conditions, or else substitute for them unknown forces. In the algebraical statement just given the former plan has been adopted; but if we choose to adopt the latter we have merely to make X, Y, Z, &c., include not only the applied forces, but also the stresses arising from the constraints.

If the Statical Principle of Virtual Velocities be employed, we have for D'Alembert's Principle the concise mode of expression given by Lagrange, and applied by him with such success in his *Mécanique Analytique*, viz. :—

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0. \quad (1)$$

202. **D'Alembert's Principle for Impulses.**—As has been stated already in Article 65, an Impulsive or Instantaneous Force is a force which produces a finite change of velocity in a time so short, that in it no sensible change of velocity is produced by the action of the forces which are not impulsive. If the constraints and connections of a system be regarded as giving rise to forces, these forces may be impulsive or not according to the nature of the constraint. For example, a blow given to a body which is resting on an immovable surface produces an impulsive reaction, provided the blow is not tangential to the surface; but a sudden jerk at the end of an extensible elastic string produces no impulsive reaction. It is important to observe that each point of the system may be regarded as occupying the same position in space at the end as at the beginning of the time during which the impulsive forces have acted. In other words, the velocities of the various points may change by a finite amount, but the positions can only change by an infinitely small amount during the time under consideration.

If $\dot{x}$, $\dot{y}$, $\dot{z}$ be the *changes* of the components of the velocity of any point, whose coordinates are x, y, z, due to the action of the impulsive forces, and, if $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$ be the components of the impulse which has acted at this point, D'Alembert's Principle as applied to impulsive forces may be expressed in the form—

$$\Sigma\{(\underset{\cdot}{X} - m\dot{x})\,\delta x + (\underset{\cdot}{Y} - m\dot{y})\,\delta y + (\underset{\cdot}{Z} - m\dot{z})\,\delta z\} = 0. \quad (2)$$

The truth of the principle in the present case can be established by reasoning similar to that employed in the preceding Article.

It may also be derived from the principle applied to continuous forces by considering the impulsive forces as continuous forces of great magnitude acting for a short time. In fact, if we multiply the equation

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0$$

by dt, and integrate between the limits t and t'; if the interval $t - t'$ be sufficiently short, the system has not sensibly altered its position, and therefore δx, &c. are the same at the end of the time as at the beginning, and we have

$$\Sigma\left\{\left[\int_{t'}^{t} Xdt - m\left(\frac{dx}{dt} - \left(\frac{dx}{dt}\right)'\right)\right]\delta x + \&\text{c.}\right\} = 0.$$

Now, if X be the component of a continuous force, $\int_{t'}^{t} Xdt$ is insensible; and, if X be the component of an impulsive force, $\int_{t'}^{t} Xdt$ is the component of the impulse along the axis of x, which may be denoted by $\underset{\cdot}{X}$; hence, as

$$\frac{dx}{dt} - \left(\frac{dx}{dt}\right)' = \dot{x}, \ \&\text{c.},$$

we obtain

$$\Sigma\{(\underset{\cdot}{X} - m\dot{x})\,\delta x + (\underset{\cdot}{Y} - m\dot{y})\,dy + (\underset{\cdot}{Z} - m\dot{z})\,\delta z\} = 0,$$

as before.

203. **Equations of Motion of a Rigid Body.**—By means of D'Alembert's Principle we can at once write down the equations of motion of a rigid body. We have, in fact, merely to write down the six equations of equilibrium, taking into account, not only the applied forces, but also the forces of inertia as defined in Art. 201.

Hence the six equations of motion are—

$$\left.\begin{aligned}
&\Sigma m\frac{d^2x}{dt^2} = \Sigma X, \quad \Sigma m\frac{d^2y}{dt^2} = \Sigma Y, \quad \Sigma m\frac{d^2z}{dt^2} = \Sigma Z \\
&\Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \Sigma(xY - yX) = N \\
&\Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \Sigma(yZ - zY) = L \\
&\Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) = \Sigma(zX - xZ) = M
\end{aligned}\right\}, \qquad (3)$$

where L, M, N are the moments of the applied forces round the axes.

For impulses, the corresponding equations are—

$$\left.\begin{aligned}
&\Sigma m\dot{x} = \Sigma \underset{\cdot}{X}, \quad \Sigma m\dot{y} = \Sigma \underset{\cdot}{Y}, \quad \Sigma m\dot{z} = \Sigma \underset{\cdot}{Z} \\
&\Sigma m(x\dot{y} - y\dot{x}) = \Sigma(x\underset{\cdot}{Y} - y\underset{\cdot}{X}) = \underset{\cdot}{N} \\
&\Sigma m(y\dot{z} - z\dot{y}) = \Sigma(y\underset{\cdot}{Z} - z\underset{\cdot}{Y}) = \underset{\cdot}{L} \\
&\Sigma m(z\dot{x} - x\dot{z}) = \Sigma(z\underset{\cdot}{X} - x\underset{\cdot}{Z}) = \underset{\cdot}{M}
\end{aligned}\right\}. \qquad (4)$$

These equations determine the motion of a rigid body which is altogether free, *i.e.* unacted on by any constraints, or geometrical conditions.

In the case of a body subject to constraints, if all six equations are employed, the constraints must be replaced by the stresses to which they give rise.

The first three equations of each group may be put into a simpler shape as follows:—

204. **Motion of the Centre of Inertia of a Free Body.**—Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of inertia, and $\mathfrak{M}$ the mass of the body; then, for continuous forces,

$$\left.\begin{aligned}
\mathfrak{M}\frac{d^2\bar{x}}{dt^2} &= \Sigma m\frac{d^2x}{dt^2} = \Sigma X \\
\mathfrak{M}\frac{d^2\bar{y}}{dt^2} &= \Sigma m\frac{d^2y}{dt^2} = \Sigma Y \\
\mathfrak{M}\frac{d^2\bar{z}}{dt^2} &= \Sigma m\frac{d^2z}{dt^2} = \Sigma Z
\end{aligned}\right\}. \qquad (5)$$

Also, for impulses,

$$\left.\begin{aligned} \mathfrak{M}\dot{\bar{x}} &= \Sigma m\dot{x} = \Sigma \underset{\cdot}{X} \\ \mathfrak{M}\dot{\bar{y}} &= \Sigma m\dot{y} = \Sigma \underset{\cdot}{Y} \\ \mathfrak{M}\dot{\bar{z}} &= \Sigma m\dot{z} = \Sigma \underset{\cdot}{Z} \end{aligned}\right\}. \qquad (6)$$

These equations give the motion of the centre of inertia of a body which is free and acted on by any forces. From them it appears that—

The centre of inertia of a free body moves as if all the forces were applied to the entire mass concentrated there.

This is a particular case of a more general principle which will be given further on.

205. **Motion of a Free Body relative to its Centre of Inertia.**—If we call ξ, η, ζ the coordinates of any point of the body referred to its centre of inertia as origin,

$$x = \bar{x} + \xi, \quad y = \bar{y} + \eta, \quad z = \bar{z} + \zeta.$$

Introducing these values into the three latter equations of groups (3) and (4), we have

$$\Sigma m \left\{(\bar{x} + \xi)\,\frac{d^2(\bar{y} + \eta)}{dt^2} - (\bar{y} + \eta)\,\frac{d^2(\bar{x} + \xi)}{dt^2}\right\} \\ = \Sigma\left\{(\bar{x} + \xi)\,Y - (\bar{y} + \eta)\,X\right\},$$

or

$$\mathfrak{M}\left(\bar{x}\frac{d^2\bar{y}}{dt^2} - \bar{y}\frac{d^2\bar{x}}{dt^2}\right) + \Sigma m\left(\xi\frac{d^2\eta}{dt^2} - \eta\frac{d^2\xi}{dt^2}\right) \\ = (\bar{x}\Sigma Y - \bar{y}\Sigma X) + \Sigma(\xi Y - \eta X),$$

whence, in virtue of the equations for the motion of the centre of inertia,

$$\left.\begin{aligned} \Sigma m\left(\xi\frac{d^2\eta}{dt^2} - \eta\frac{d^2\xi}{dt^2}\right) &= \Sigma(\xi Y - \eta X) \\ \Sigma m\left(\eta\frac{d^2\zeta}{dt^2} - \zeta\frac{d^2\eta}{dt^2}\right) &= \Sigma(\eta Z - \zeta Y) \\ \Sigma m\left(\zeta\frac{d^2\xi}{dt^2} - \xi\frac{d^2\zeta}{dt^2}\right) &= \Sigma(\zeta X - \xi Z) \end{aligned}\right\}, \qquad (7)$$

and there are three corresponding equations for the impulsive forces. These equations show that

The motion of a free body relative to its centre of inertia is the same as if this point were fixed in space, the applied forces being absolutely unaltered as regards magnitude, direction, and point of application.

The general problem of the motion of a rigid body has now been reduced to the determination of the motion of the centre of inertia, and of the motion of the body relative to its centre of inertia. Thus the general problem of the motion of the Earth involves the consideration of the path described by its centre of inertia under the attraction of the sun and planets, and of the rotatory motion of the Earth round its centre regarded as fixed. The former problem belongs to the Planetary Theory. The solution of the latter problem forms the Theory of Precession and Nutation.

206. **Moments of Momentum.**—The equation

$$\Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \Sigma(xY - yX),$$

in its primary signification, expresses that the moment of the forces of inertia round the axis of z is equal and opposite to the moment of the applied forces.

Since $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right),$$

there are two other modes in which this equation may be interpreted.

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

is the moment of the momenta of the elements of the body round the axis of z, and the equation in question asserts that—

The change in the moment of the momenta of the elements of the body round the axis of z, during an indefinitely short time, divided by the time, is equal to the moment of the applied forces.

It may also, however, be regarded as expressing that, the change in twice the sum of the elementary areas described round the origin O, by the projections of the points of the body on a fixed plane through O, multiplied each by the corresponding element of mass, is equal to the moment of the applied forces round the normal to the plane of the areas, multiplied by the element of the time.

In fact, if $\frac{1}{2}h_3dt$ be the elementary area described round the origin by the projection of the point x, y, z on the plane of xy, then

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h_3;$$

hence the equation becomes

$$\Sigma m\frac{dh_3}{dt} = \Sigma(xY - yX).$$

If, as before, we denote the moments of the applied forces round the axes by L, M, N, the foregoing equation and the two others corresponding become

$$\Sigma m\frac{dh_1}{dt} = L, \quad \Sigma m\frac{dh_2}{dt} = M, \quad \Sigma m\frac{dh_3}{dt} = N. \tag{8}$$

The corresponding equations for impulses are

$$\Sigma mh_1 = \underset{\cdot}{L}, \quad \Sigma mh_2 = \underset{\cdot}{M}, \quad \Sigma mh_3 = \underset{\cdot}{N}. \tag{9}$$

If the body is in motion when the impulses act, the three latter equations should be written

$$\Sigma mh_1 = \underset{\cdot}{L} + \Sigma mh'_1, \quad \Sigma mh_2 = \underset{\cdot}{M} + \Sigma mh'_2, \quad \Sigma mh_3 = \underset{\cdot}{N} + \Sigma mh'_3, \tag{10}$$

where h'_1, h'_2, h'_3 are the values of h_1, h_2, h_3 the instant before the impulses act.

The quantities h_1, h_2, h_3, &c. are twice the areal velocities, relative to the origin, of the different points of the body; and $\frac{dh_1}{dt}$, &c. are twice the areal accelerations (*see* Art. 29).

As the areal equations for a rigid body contain an infinite number of accelerations, the left-hand members being in fact definite integrals, it is necessary to transform them before they can be applied to any particular question.

207. **The General Equations of Motion of a Rigid Body apply to every System.**—It is well known that the forces acting on any system in equilibrium must fulfil the conditions which would be required if the system were rigid. Hence the equations of motion of a rigid body are, in their most general form, applicable to any system whatever. The special reductions which may be applied to the forces of inertia in the case of a rigid body cannot, however, be employed in other cases.

208. **Equation of Vis Viva.**—A first integral of the equations of motion can very frequently be obtained directly from D'Alembert's Principle, as follows.—

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0,$$

where δx, δy, δz, &c. are arbitrary displacements consistent with the conditions of the system. If the equations of condition do not contain the time explicitly, dx (the actual movement of the point along the axis of x during an infinitely short time) is always a value which may be legitimately assigned to δx; for the fact that it is an actual displacement shows that it is consistent with the equations of condition, and therefore possible, provided these equations do not alter with the time, that is, do not contain the time explicitly. If they contain the time explicitly, dx is not in general a possible value of δx. In fact, $U = 0$ being an equation of condition, if U is a function of the coordinates simply, δx and dx must satisfy the same equation, viz.,

$$\frac{dU}{dx}\delta x + \frac{dU}{dy}\delta y + \&c. = 0.$$

If, however, U contains t explicitly, δx has to satisfy the equation

$$\frac{dU}{dx}\delta x + \&c. = 0,$$

where t is treated as constant; but dx has to satisfy the equation

$$\frac{dU}{dx}dx + \&c. + \frac{dU}{dt}dt = 0.$$

This is so, because dx is the interval between two *successive* positions of a point, at consecutive instants of time, whereas δx is the interval between two *simultaneous* infinitely near possible positions of the point.

In the great majority of problems dx is a possible value of δx; and the same holds for the other displacements. Assuming then that the transformation is legitimate, let us assign to δx, δy, δz, &c. the values dx, dy, &c.; D'Alembert's equation becomes then

$$\Sigma m\left(\frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy + \frac{d^2z}{dt^2}\,dz\right) = \Sigma\,(Xdx + Ydy + Zdz).$$

Integrating, we have

$$\Sigma m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = 2\Sigma\int(Xdx + Ydy + Zdz) + c, \quad (11)$$

where c is an arbitrary constant.

This equation is called the equation of *vis viva.*

If we denote the vis viva at any particular time t' by $\Sigma mv'^2$, the equation above may be written

$$\Sigma mv^2 - \Sigma mv'^2 = 2\Sigma\int(Xdx + Ydy + Zdz), \qquad (12)$$

where the right-hand side is twice the work done by the forces in passing from the position occupied at the time t' to the position occupied at the time t.

The equation of vis viva for a rigid body has been already obtained in a different manner in Article 129.

The equation of vis viva is one of the most important in Dynamics. It is to a great extent the foundation of the Theory of Energy, and is the basis of the celebrated transformations of Lagrange. This equation will be more fully considered in a future chapter.

209. **Of the Forces which enter the Equation of Vis Viva.**—It is plain from the mode in which the equation of vis viva has been deduced from D'Alembert's Principle that in the case of a rigid body the right-hand side contains only the applied forces, and that reactions by which geometrical conditions may be replaced do not enter therein. The reactions of fixed points, fixed surfaces, &c., are thus excluded: but further, if during the motion the direction of a

force be at each instant at right angles to the line in which its point of application is moving, such a force does not enter the equation of vis viva (Art. 121).

When two surfaces roll on one another without slipping, the relative tangential velocity is zero. Now, if F be the tangential force of friction developed between them, the element of work done by F on one body is Fdf_1, and that done on the other is $-Fdf_2$, df_1 and df_2 being the projections on the direction of F of the small motions of the two points in contact. Hence the whole work done by the tangential friction is $F(df_1 - df_2)$; but $\frac{df_1}{dt} - \frac{df_2}{dt}$ is the relative tangential velocity of the points of the surfaces which are in contact. Hence the whole work done by the tangential force of friction in *pure rolling* is zero, or this force in the case supposed has no effect on the equation of vis viva. If one of the surfaces be fixed, a similar result can be arrived at with still greater facility.

When two surfaces are in permanent contact, the normal reaction between them never enters the equation of vis viva. In this case, if the motion be pure slipping, the relative velocity of the points of application of the mutual reaction is altogether tangential. If the motion be either rolling and slipping, or pure rolling, the relative normal velocity of the points in contact is still zero, or at least infinitely small, and the relative normal displacement is an infinitely small quantity of the second order, that is, $dr_1 - dr_2 = 0$, and therefore $R(dr_1 - dr_2)$, which is the whole work done by the mutual normal reaction, is equal to zero.

210. **Effect of Collision on Vis Viva.**—The work done in the element of time dt, by the mutual normal reaction of two bodies which collide, is always of the form $R(dr_1 - dr_2)$, where $\frac{dr_1}{dt} - \frac{dr_2}{dt}$ is the relative normal velocity. dt is here, as in Art. 202, an infinitely small quantity of the second order. If we call r the projection on the common normal of the distance between the points of the bodies which act on one another, r is an infinitely small quantity of the first order, and the whole work done by the mutual action in the element of time dt is Rdr.

If the mutual force R between two bodies tends to increase the relative velocity which already exists in the direction of r, Rdr is positive. If on the other hand R tends to diminish the relative normal velocity which already exists, Rdr is negative. This readily appears from the following considerations:—

If the mutual action tends to increase $\frac{dr_1}{dt}$, it tends to diminish $\frac{dr_2}{dt}$, and therefore the element of work done by it is $R(dr_1 - dr_2)$, which is positive if $\frac{dr_1}{dt} > \frac{dr_2}{dt}$, and negative if $\frac{dr_2}{dt} > \frac{dr_1}{dt}$. In the first case the mutual action tends to increase the relative normal velocity; and in the second case to diminish this velocity. Also, if the mutual action tends to diminish $\frac{dr_1}{dt}$ similar reasoning applies. Hence we may conclude that in the case of collision between two bodies the work done by the mutual normal reaction is negative while this reaction is diminishing the relative normal velocity already existing, and positive while it is increasing the existing relative normal velocity. In the first case, the vis viva of the bodies is diminished, and, in the second, increased.

As has been already stated (Art: 76), in the impact of bodies there are two periods. In the first, the mutual action reduces the relative normal velocity of the colliding points to zero. In the second, a force of restitution is developed which acts on each body in the same direction as the original force. In the first period, as the mutual action reduces the relative normal velocity to zero, and consequently diminishes its previously existing amount, the vis viva is diminished. In the second period the vis viva is increased, as the force of restitution generates a new relative normal velocity, and continually increases the relative velocity already produced.

If the bodies which collide be perfectly elastic, R (the mutual normal action) is a function of r, both in the period of compression and likewise in the period of restitution.

Hence $R\,dr$ is a perfect differential. Moreover, in the case of perfect elasticity, the mutual distances between the points of the bodies affected by the shock regain at the end of the shock precisely the same values which they had at the beginning. Hence the complete value of $\int R\,dr$, taken through the two periods of compression and restitution, is, in the case of perfectly elastic bodies, zero, and therefore vis viva is unaffected by the shock of perfectly elastic bodies.

If the bodies which collide be imperfectly elastic, vis viva is lost in the period of compression, and gained in the period of restitution; but the amount gained in the second period is less than that lost in the first, so on the whole, in the shock of imperfectly elastic bodies vis viva is lost (*see* Art. 135, Ex. 13).

SECTION III.—*Motion round a Fixed Axis.*

211. **Special Cases of Motion. Degrees of Freedom.**—In order to transform the general equations of motion in such a way as to be of use in particular problems, it is necessary to know something of the special conditions of the problem which it is required to solve.

We have seen in Article 190 that six conditions are required to fix the position of a rigid body, and we have found accordingly six equations of motion for a body perfectly free. Such a body is said to have *six degrees of freedom* (Art. 191). We have obtained the equations for this case in their most general form, but we shall now adopt the reverse method of procedure, and consider the special equations to be employed for a body having one degree of freedom.

212. **One Degree of Freedom.**—A body is said to have one degree of freedom when its position is limited in such a way as to depend on a single indeterminate quantity. It will be shown subsequently that the variations of the coordinates of any point of a body entirely free are linear functions of six undetermined quantities. If these six quantities are connected together in such a way that one being given

all the rest are determined, the body has one degree of freedom.

The simplest cases of one degree of freedom occur when some of the six undetermined displacements are zero. We shall consider here only two cases.

(1) If the motion of the body be limited to a series of pure translations, and the path of one of its points be assigned.

(2) If the motion of the body be limited to a rotation round an axis fixed in space.

In the first case the problem is readily reducible to that of the constrained motion of a particle.

This reduction is most easily effected by employing D'Alembert's Principle as expressed by Lagrange. In fact we have

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0.$$

Now, by the conditions of the question δx, δy, δz must be the same for every point of the body, and ds being the arc of the curve described by the centre of inertia,

$$\delta x = \frac{d\bar{x}}{ds}\,\delta s, \quad \delta y = \frac{d\bar{y}}{ds}\,\delta s, \quad \delta z = \frac{d\bar{z}}{ds}\,\delta s.$$

Making these substitutions, we obtain the single equation of motion,

$$\left(\Sigma m\frac{d^2x}{dt^2}\right)\frac{d\bar{x}}{ds} + \left(\Sigma m\frac{d^2y}{dt^2}\right)\frac{d\bar{y}}{ds} + \left(\Sigma m\frac{d^2z}{dt^2}\right)\frac{d\bar{z}}{ds}$$

$$= (\Sigma X)\frac{d\bar{x}}{ds} + (\Sigma Y)\frac{d\bar{y}}{ds} + (\Sigma Z)\frac{dz}{ds};$$

or, as $$ds^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2,$$

we have finally, if we put $\mathfrak{M}$ for the whole mass of the body,

$$\mathfrak{M}\frac{d^2s}{dt^2} = S, \qquad (1)$$

where S is the sum of the components of all the applied forces along the tangent to the path of the centre of inertia; but this is obviously the equation required for determining the constrained motion of a particle.

213. **To Determine the Motion of a Body constrained to move round an Axis fixed in Space.**—Take the fixed axis as axis of x. The condition of equilibrium for a body having a fixed axis is that the moment of the forces round the axis should be zero. By D'Alembert's Principle the equation of motion is, therefore,

$$\Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \Sigma\,(yZ - zY) = L.$$

If p be the perpendicular from the point x, y, z on the axis of x, and θ the angle which p makes with the axis of y, we have $y = p\cos\theta$, $z = p\sin\theta$; whence, putting $\omega = \frac{d\theta}{dt}$,

$$\Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \Sigma m\frac{d}{dt}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$= \Sigma m\frac{d}{dt}\,(p^2\omega) = \frac{d\omega}{dt}\,\Sigma mp^2,$$

as the p for each point is invariable.

Since Σmp^2 is the moment of inertia of the body round the fixed axis (*Integral Calculus*, Chap. x.), if k be the radius of gyration of the body, and $\mathfrak{M}$ its mass, we have,

$$\mathfrak{M}k^2\frac{d\omega}{dt} = L. \qquad (2)$$

Hence,

The angular acceleration of the body is equal to the moment of the applied forces divided by the moment of inertia.

The equation just given was obtained before in Art. 135 by a different method.

The corresponding equation for an impulse is

$$\mathfrak{M} k^2 \dot{\theta} = L, \tag{3}$$

or,

The moment of inertia multiplied by the change of the angular velocity due to the impulse is equal to the moment of the impulse.

214. **Equation of Vis Viva for a Body moving round a Fixed Axis.**—The expression for the vis viva of a body moving round a fixed axis has been given already, Art. 130. If we take the fixed axis for the axis of x, we have, as the equation of vis viva,

$$\mathfrak{M} k^2 \omega^2 = 2\Sigma \int (Y dy + Z dz) + c. \tag{4}$$

215. **Stresses on the Axis of Rotation.**—We have seen that D'Alembert's Principle furnishes at once the single equation of motion which is required to determine the velocity and position of a body rotating round a fixed axis. The same principle enables us to write down the equations which are required to determine the stresses on the axis.

In order to determine these stresses, we may regard the body as compelled to rotate round the fixed axis, not by a geometrical condition, but by forces acting on the body at any two points on the axis. The body is then to be considered free, but the magnitude of the forces replacing the constraints is such as to compel the body to rotate round the given axis.

The axis of x being the axis of rotation, we have, as before, $x = \text{constant}$, $y = p\cos\theta$, $z = p\sin\theta$, whence

$$\frac{dx}{dt} = 0,\ \frac{dy}{dt} = -p\sin\theta\frac{d\theta}{dt},\quad \frac{dz}{dt} = p\cos\theta\frac{d\theta}{dt};\ \ \frac{d^2x}{dt^2} = 0,$$

$$\frac{d^2y}{dt^2} = -p\sin\theta\frac{d^2\theta}{dt^2} - p\cos\theta\left(\frac{d\theta}{dt}\right)^2 = -z\frac{d^2\theta}{dt^2} - y\left(\frac{d\theta}{dt}\right)^2,$$

$$\frac{d^2z}{dt^2} = p\cos\theta\frac{d^2\theta}{dt^2} - p\sin\theta\left(\frac{d\theta}{dt}\right)^2 = y\frac{d^2\theta}{dt^2} - z\left(\frac{d\theta}{dt}\right)^2.$$

Accordingly,

$$\Sigma m\frac{d^2x}{dt^2}=0,\quad \Sigma m\frac{d^2y}{dt^2}=-\frac{d^2\theta}{dt^2}\Sigma mz-\left(\frac{d\theta}{dt}\right)^2\Sigma my,$$

$$\Sigma m\frac{d^2z}{dt^2}=\frac{d^2\theta}{dt^2}\Sigma my-\left(\frac{d\theta}{dt}\right)^2\Sigma mz,$$

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=-\frac{d^2\theta}{dt^2}\Sigma mxz-\left(\frac{d\theta}{dt}\right)^2\Sigma mxy,$$

$$\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=\frac{d^2\theta}{dt^2}\Sigma m(y^2+z^2),$$

$$\Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)=-\frac{d^2\theta}{dt^2}\Sigma mxy+\left(\frac{d\theta}{dt}\right)^2\Sigma mxz.$$

In the present case Σmxy, Σmxz, and $\Sigma m(y^2+z^2)$ are definite integrals, being the products and moment of inertia of the body (*Integral Calculus*, Art. 195), and

$$\Sigma my=\mathfrak{M}\bar{y},\quad \Sigma mz=\mathfrak{M}\bar{z}.$$

If now X_1, Y_1, Z_1; X_2, Y_2, Z_2, acting at the arbitrary points x_1 and x_2 on the axis, be the components of the forces by which the action of the fixed axis on the body is replaced, equations (3) of Art. 203, enable us to determine

$$\theta,\quad X_1+X_2,\quad Y_1,\quad Y_2,\quad Z_1,\quad \text{and}\quad Z_2;$$

provided we can integrate the equation which gives the value of $\dfrac{d^2\theta}{dt^2}$. This equation is the same as that obtained before in Art. 213. If the same notation be adopted for the applied forces as in Art. 203; since

$$y_1=y_2=z_1=z_2=0,$$

equations (3) of Art. 203 become

$$\left.\begin{aligned}
&\Sigma X + X_1 + X_2 = 0 \\
&-\mathfrak{M}\bar{z}\frac{d^2\theta}{dt^2} - \mathfrak{M}\bar{y}\left(\frac{d\theta}{dt}\right)^2 = \Sigma Y + Y_1 + Y_2 \\
&\mathfrak{M}\bar{y}\frac{d^2\theta}{dt^2} - \mathfrak{M}\bar{z}\left(\frac{d\theta}{dt}\right)^2 = \Sigma Z + Z_1 + Z_2 \\
&-\frac{d^2\theta}{dt^2}\int xz\,dm - \left(\frac{d\theta}{dt}\right)^2\int xy\,dm = \Sigma(xY - yX) + x_1Y_1 + x_2Y_2 \\
&\frac{d^2\theta}{dt^2}\int (y^2 + z^2)\,dm = \Sigma(yZ - zY) \\
&-\frac{d^2\theta}{dt^2}\int xy\,dm + \left(\frac{d\theta}{dt}\right)^2\int xz\,dm = \Sigma(zX - xZ) - x_1Z_1 - x_2Z_2
\end{aligned}\right\}. \qquad (5)$$

Instead of regarding the reaction of the fixed axis against the body as consisting of two forces, we may consider it as equivalent to a single force passing through the origin and a couple. When θ has been found from the fifth of equations (5), the first three of those equations determine the components of the force, and the fourth and sixth those of the couple.

216. **Axis of Rotation a Principal Axis through the Centre of Inertia.**—If the axis of rotation be a *principal axis* through the centre of inertia,

$$\bar{y} = \bar{z} = 0,\ \int xy\,dm = \int xz\,dm = 0,$$

whence equations (5) become

$$\left.\begin{aligned}
&\Sigma X + X_1 + X_2 = 0,\ \Sigma Y + Y_1 + Y_2 = 0,\ \Sigma Z + Z_1 + Z_2 = 0 \\
&N + x_1Y_1 + x_2Y_2 = 0,\ \mathfrak{M}k^2\frac{d^2\theta}{dt^2} = L,\ M - x_1Z_1 - x_2Z_2 = 0
\end{aligned}\right\}. \qquad (6)$$

Five of these equations are the same as those required in order that the applied forces should be in equilibrium with the reactions of the axis against the body.

If no external forces are applied to the body,

$$X_1 + X_2 = 0,\ Y_1 + Y_2 = 0,\ Z_1 + Z_2 = 0;\ x_1Y_1 + x_2Y_2 = 0,\ x_1Z_1 + x_2Z_2 = 0,$$

and therefore

$$Y_1 = 0,\quad Y_2 = 0,\quad Z_1 = 0,\quad Z_2 = 0;\ \text{or finally—}$$

If a body be rotating round a principal axis through its centre of inertia, no forces being supposed to act, there is no stress on the axis, and the body will continue to rotate round that axis with a uniform angular velocity.

This result was obtained before in Article 96.

217. **Impulses.**—If a body, rotating round a fixed axis, taken as the axis of x, be acted on by any system of impulses, since for each point of the body x remains invariable, and $y = p\cos\theta$, $z = p\sin\theta$, where p does not vary with the time, we have

$$\dot{x} = 0,\ \dot{y} = \frac{dy}{dt} - \frac{dy_0}{dt} = -z\left(\frac{d\theta}{dt} - \frac{d\theta_0}{dt}\right) = -z\dot{\theta},$$

as the position of any point remains unaltered while the impulsive forces are acting. In like manner $\dot{z} = y\dot{\theta}$.

If we desire now to obtain the equations which give the magnitudes of the impulses exercised by the fixed axis, as well as the change of the body's angular velocity, we have only to substitute in equations (4), Article 203, the values just obtained for $\dot{x}$, $\dot{y}$, and $\dot{z}$. These equations become, then,

$$\left.\begin{aligned}
&\Sigma\underset{\cdot}{X} + \underset{\cdot}{X}_1 + \underset{\cdot}{X}_2 = 0\\
&-\mathfrak{M}\bar{z}\,\dot{\theta} = \Sigma\underset{\cdot}{Y} + \underset{\cdot}{Y}_1 + \underset{\cdot}{Y}_2\\
&\mathfrak{M}\bar{y}\,\dot{\theta} = \Sigma\underset{\cdot}{Z} + \underset{\cdot}{Z}_1 + \underset{\cdot}{Z}_2\\
&-\dot{\theta}\int xz\,dm = \Sigma(x\underset{\cdot}{Y} - y\underset{\cdot}{X}) + x_1\underset{\cdot}{Y}_1 + x_2\underset{\cdot}{Y}_2\\
&\mathfrak{M}(k^2 + \bar{p}^2)\,\dot{\theta} = \Sigma(y\underset{\cdot}{Z} - z\underset{\cdot}{Y})\\
&-\dot{\theta}\int xy\,dm = \Sigma(z\underset{\cdot}{X} - x\underset{\cdot}{Z}) - x_1\underset{\cdot}{Z}_1 - x_2\underset{\cdot}{Z}_2
\end{aligned}\right\},\qquad (7)$$

where $\underset{\cdot}{X}_1$, $\underset{\cdot}{Y}_1$, $\underset{\cdot}{Z}_1$, $\underset{\cdot}{X}_2$, $\underset{\cdot}{Y}_2$, $\underset{\cdot}{Z}_2$ are the components of the impulses exercised at two points by the axis.

218. **Centre of Percussion.**—If a body receive a blow which makes it begin to rotate round a fixed axis, without causing any impulsive pressure on the axis, the point in which the direction of the blow intersects the plane containing the fixed axis and the centre of inertia is called the centre of percussion. In order that such a point should exist, both

the axis and the line of direction of the impulse must fulfil certain conditions, which we proceed to investigate.

In this case, by hypothesis, $\underset{\cdot}{X_1} = \underset{\cdot}{X_2} = \underset{\cdot}{Y_1} = \underset{\cdot}{Y_2} = \underset{\cdot}{Z_1} = \underset{\cdot}{Z_2} = 0$. If we call the components of the blow $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$, and the coordinates of any point in its line of direction x', y', z', we have

$$\left.\begin{array}{c} \underset{\cdot}{X} = 0, \quad -\mathfrak{M}\bar{z}\dot{\theta} = \underset{\cdot}{Y}, \quad \mathfrak{M}y\dot{\theta} = \underset{\cdot}{Z}, \\ -\dot{\theta}\int xzdm = x'\underset{\cdot}{Y}, \quad -\dot{\theta}\int xydm = -x'\underset{\cdot}{Z}, \\ \mathfrak{M}(k^2 + \bar{p}^2)\,\dot{\theta} = y'\underset{\cdot}{Z} - z'\underset{\cdot}{Y}. \end{array}\right\} \qquad (8)$$

Since $\underset{\cdot}{X} = 0$, the direction of the blow must be at right angles to the fixed axis, hence it may be supposed to act in the plane of yz, and to be parallel to the axis of y: this gives $x' = 0$, and $\underset{\cdot}{Z} = 0$. Hence $\bar{y} = 0$, or *the centre of inertia must lie in a plane through the fixed axis, at right angles to the direction of the impulse.* Again, as $x' = 0$, we have $\int xz dm = 0$, and $\int xy dm = 0$; hence *the axis must be a principal axis for the point in which it is met by its shortest distance from the line of direction of the impulse.* Finally we have

$$\frac{k^2 + \bar{p}^2}{z'} = \bar{z}, \quad \text{or} \quad z' = \frac{k^2 + \bar{p}^2}{\bar{z}};$$

but, as $\bar{y} = 0$, $\bar{z} = \bar{p}$, the distance of the centre of inertia from the fixed axis, whence

$$z' = \frac{k^2 + \bar{p}^2}{\bar{p}}. \qquad (9)$$

Hence *the distance of the centre of percussion from the fixed axis is the same as that of the centre of oscillation* (Art. 133).

Moreover, if ξ, η, ζ be the coordinates of any point relatively to the centre of inertia,

$$\int xz\, dm = \mathfrak{M}\bar{x}\bar{z} + \int \xi\zeta\, dm;$$

whence, if the axis be parallel to a principal axis through the centre of inertia, $\bar{x} = 0$, and *the shortest distance between the direction of the blow and the fixed axis passes through the centre of inertia, and the centre of percussion coincides with the centre of oscillation.*

EXAMPLES.

1. A heavy body starting from rest turns under the action of gravity round a fixed horizontal axis, which is a principal axis at the centre of suspension. Find the stress on the axis.

Take the centre of suspension (Art. 133) for origin, and the fixed axis for that of x. Let the axis of y be horizontal, and θ the angle made with it at any instant by the line joining the centre of inertia to the centre of suspension. Let p be the length of this line, and take the positive direction of z upwards, then

$$\bar{x} = 0, \quad \bar{y} = p \cos\theta, \quad \bar{z} = p \sin\theta, \quad \int xy\,dm = \int xz\,dm = 0.$$

Equations (5), Article 215, give

$$X_1 + X_2 = 0, \quad x_1 Y_1 + x_2 Y_2 = 0, \quad x_1 Z_1 + x_2 Z_2 = 0,$$

$$\frac{d^2\theta}{dt^2} = -\frac{gp}{k^2+p^2}\cos\theta, \text{ and therefore } \left(\frac{d\theta}{dt}\right)^2 = \frac{2gp}{k^2+p^2}(\sin\alpha - \sin\theta),$$

where α is the initial value of θ; whence, substituting in the remaining two of those equations, we get

$$Y_1 + Y_2 = \mathfrak{M}g\,\frac{p^2}{k^2+p^2}\cos\theta\,(3\sin\theta - 2\sin\alpha).$$

$$Z_1 + Z_2 = \mathfrak{M}g\left\{\frac{k^2}{k^2+p^2} + \frac{p^2}{k^2+p^2}\sin\theta\,(3\sin\theta - 2\sin\alpha)\right\}.$$

Hence the forces exerted by the axis on the body consist of a single force passing through the centre of suspension whose components are $Y_1 + Y_2$, and $Z_1 + Z_2$.

The stress on the axis is equal and opposite. The components of this stress, along, and at right angles to, the line joining the centre of suspension to the centre of inertia, are

$$-\mathfrak{M}g\left\{\frac{k^2+3p^2}{k^2+p^2}\sin\theta - \frac{2p^2}{k^2+p^2}\sin\alpha\right\},$$

and

$$-\mathfrak{M}g\,\frac{k^2}{k^2+p^2}\cos\theta.$$

When the centre of inertia is below the axis of suspension, θ is negative, and these stresses are both downwards if α be positive.

2. To the ends of a thin light piece of wood are fastened spheres of lead whose weights are P and P'. The piece of wood turns on a horizontal axis through its middle point. Its length being $2l$, and its mass negligible, determine the time of a small oscillation, the spheres being so small that the squares of their radii are negligible as compared with l.

Ans. $$\pi\sqrt{\frac{l}{g}}\cdot\sqrt{\frac{P+P'}{P-P'}}.$$

By changing P, and comparing the times of oscillation, an apparatus of the kind mentioned can be used to verify the Laws of Motion.

3. A heavy pendulum, capable of revolving round a horizontal axis, is struck when at rest by a bullet moving in a horizontal direction at right angles to the fixed axis. The bullet remains in the pendulum. If b be the distance of the extremity of the pendulum from the axis, c the distance traversed by that extremity under the influence of the shot, a the distance from the axis at which the bullet penetrates, v the velocity of the bullet at impact, m its mass, M that of the pendulum, k its radius of gyration round the fixed axis, and p the distance of the latter from the centre of inertia; prove that

$$v = \frac{c}{mab}\sqrt{\{g[Mk^2 + ma^2](Mp + ma)\}}.$$

A pendulum such as that described above is called a Ballistic Pendulum. It has been employed by numerous Physicists to determine the velocity of bullets.

For a detailed description of it, and of the experiments made by its means, see Routh, *Rigid Dynamics*, Arts. 121–125.

4. A rigid body is turning round a fixed axis under the influence of a couple, whose axis is parallel to the axis of rotation: what condition must be fulfilled in order that the axis should suffer a pressure at only one point? (Schell, *Theorie der Bewegung und der Kräfte*).

The axis of rotation must be a principal axis at this point. The pressure is then at right angles to the axis. These results readily appear from the equations of Art. 215, by making the origin the point where pressure is permitted.

5. If the pressure at the fixed point vanishes, what further condition must be fulfilled?

The point must be the centre of inertia.

6. In Atwood's machine, if the pulley be not perfectly rough, and slipping takes place, determine the motion: the weight of the rope and the friction of the pulley on the axle being neglected.

If an acceleration equal and opposite to that by which it is actually animated were applied to each element of the string it would be in equilibrium; but the mass of the string being negligible, the force corresponding to this acceleration is zero q.p. Hence the other forces acting on the element of the string are in equilibrium, and μ being the coefficient of friction, and T, T' the tensions of the two ends of the rope (Minchin, *Statics*), $T' = Te^{-\mu\pi} = \lambda T$.

If z be the height from the ground of the ascending weight W', M the mass of the pulley, k its radius of gyration, a its radius, θ the angle through which it has turned, we have also

$$\frac{T' - W'}{W'}g = \frac{d^2z}{dt^2} = \frac{W - T}{W}g,$$

$$Mk^2\frac{d^2\theta}{dt^2} = a(T - T').$$

If the pulley be homogeneous, $k^2 = \frac{a^2}{2}$, and we have finally,

$$T = \frac{2WW'}{\lambda W + W'}, \quad \frac{d^2z}{dt^2} = \frac{\lambda W - W'}{\lambda W + W'}g,$$

$$a\frac{d^2\theta}{dt^2} = 4(1 - \lambda)\frac{WW'}{M(\lambda W + W')}$$

7. Taking into account the friction on the axle, and supposing the outside of the pulley to be perfectly rough, and the inside to slip on the axle, determine the motion.

The mass of the string being neglected, we may, as in the last example, regard it as acted on by a system of forces in equilibrium. Hence (as this equilibrium would not be disturbed if the string were rigid) the tensions T and T' at its extremities must equilibrate the pressure and friction exerted by the pulley against the string; and, conversely, T and T' must be equivalent to the pressure and friction exerted by the string against the pulley. Hence we may consider the pulley as acted on by the forces T, T', and its own weight; and we have, by Art. 216, if Q be the pressure on the axle, $Q = T + T' + Mg$. The moment of the couple resulting from the friction is proportional to Q, and may be written in the form βQ, where β is the product of the radius of the axle and the coefficient of friction relative to it. Substituting for the equation

$$Mk^2 \frac{d^2\theta}{dt^2} = a(T - T')$$

of Ex. 6, the equation

$$Mk^2 \frac{d^2\theta}{dt^2} = a(T - T') - \beta(T + T' + Mg);$$

and remembering that as the pulley is perfectly rough, $a\frac{d\theta}{dt} = \frac{dz}{dt}$, we obtain if we put $\nu = \frac{\beta}{a}$,

$$T = \frac{(1 + 2\nu)Mg + 4(1 + \nu)W'}{Mg + 2(1 - \nu)W + 2(1 + \nu)W'} \cdot W.$$

$$T' = \frac{(1 - 2\nu)Mg + 4(1 - \nu)W}{Mg + 2(1 - \nu)W + 2(1 + \nu)W'} \cdot W'.$$

$$\frac{d^2z}{dt^2} = \frac{(1 - \nu)W - (1 + \nu)W' - \nu Mg}{(1 - \nu)W + (1 + \nu)W' + \frac{1}{2}Mg} \cdot g.$$

8. If the pulley be not perfectly rough, and slipping of the string on the pulley takes place, determine the motion, taking into account the friction on the axle, and supposing the inside of the pulley to slip as before.

In this case, as in Ex. 6, the acceleration of the weights is quite independent of the mass and size of the pulley, and we have

$$T = \frac{2WW'}{\lambda W + W'}, \quad T' = \lambda T, \quad \frac{d^2z}{dt^2} = \frac{\lambda W - W'}{\lambda W + W'} \cdot g.$$

$$a\frac{d^2\theta}{dt^2} = \left\{\frac{4\{1 - \nu - \lambda(1 + \nu)\}WW'}{Mg(\lambda W + W')} - 2\nu\right\} g.$$

9. A uniform heavy rod hangs from a horizontal pivot passing through one of its extremities. An inextensible string, whose weight is negligible, attached

to the other extremity, passes through a smooth ring situated on the vertical line through the pivot at a distance below it equal to the length of the rod, and sustains a weight. The rod being slightly displaced from its position of equilibrium, determine the motion.

The equations of motion are

$$\frac{4}{3}ma^2\frac{d^2\theta}{dt^2} = -mga\sin\theta - 2aT, \quad p\frac{d^2z}{dt^2} = T - pg,$$

where a is half the length of the rod, m its mass, z the vertical coordinate of the weight, and p its mass. If z be measured from the position of the weight when the rod is vertical, $z = 4a\sin\frac{1}{2}\theta$. Since θ is always small, we may take $\sin\theta = \theta$; differentiating and eliminating T, we have, then,

$$\frac{4}{3}a(m+3p)\frac{d^2\theta}{dt^2} = -mg\left(\theta + \frac{2p}{m}\right).$$

Hence the rod completes its swing in a time

$$2\sqrt{\left\{\frac{4a}{3g}\left(1+\frac{3p}{m}\right)\right\}}\cos^{-1}\frac{2p}{2p+m\theta_0},$$

where θ_0 is the initial value of θ.

Section IV.—*Free Motion Parallel to a Fixed Plane—Kinetics.*

219. **Equations of Motion.**—The motion of a body relative to its centre of inertia consists at any instant of a velocity of rotation round some axis through the centre of inertia. Moreover, this axis must be at right angles to the fixed plane, and is, therefore, fixed in space if the position of the centre of inertia be regarded as invariable. Now, by Art. 205, the motion relative to the centre of inertia is the same as if that point were fixed in space, the forces remaining unaltered. Hence, taking the plane of yz for the fixed plane, we have, to determine the motion of the body, the three equations,

$$\mathfrak{M}\frac{d^2\bar{y}}{dt^2} = \Sigma Y, \quad \mathfrak{M}\frac{d^2\bar{z}}{dt^2} = \Sigma Z, \quad \mathfrak{M}k^2\frac{d^2\theta}{dt^2} = L, \qquad (1)$$

where $\bar{y}$ and $\bar{z}$ are the coordinates of the centre of inertia, k the radius of gyration round an axis through it at right angles to the fixed plane, and L the moment of the applied forces.

If the axis of rotation through the centre of inertia be always parallel to a line fixed in space, it is plain that the last of these equations holds good no matter whether the whole motion of the body be parallel to a fixed plane or not. In the latter case the only difference will be that a third equation, viz.,

$$\mathfrak{M}\frac{d^2\overline{x}}{dt^2} = \Sigma X,$$

will be required to determine the motion of the centre of inertia. In any case, therefore, the motion of the body is determined, when we know the motion of its centre of inertia, and the angular motion relative to that point.

220. **Connexion of the Angular Velocity with the Velocity of the Centre of Inertia.**—As the motion is parallel to a fixed plane, the parallel section of the body passing through the centre of inertia must at each instant be rotating round a point in its own plane (Art. 195). If ρ be the distance from this point (the instantaneous centre of rotation) to the centre of inertia, s the path of the latter, and ω the angular velocity, then $\rho\omega = \frac{ds}{dt}$, as is obvious. Also $\omega = \frac{d\theta}{dt}$.

221. **Equation of Vis Viva.**—It was shown in Art. 131, that the vis viva of any system $\Sigma mv^2 = \mathfrak{M}V^2 + \Sigma mv'^2$, where $\mathfrak{M}$ is the entire mass of the system, V the velocity of its centre of inertia, and v' the velocity, relative to the centre of inertia, of any particle of the system whose mass is m. If the body be moving parallel to a fixed plane, the motion relative to the centre of inertia is a rotation round an axis fixed in the body, whose direction is fixed in space. Hence $\Sigma mv'^2 = \mathfrak{M}k^2\omega^2$ (Art. 130), and the equation of vis viva becomes

$$\mathfrak{M}(V^2 + k^2\omega^2) = 2\Sigma\int(Ydy + Zdz) + C. \qquad (2)$$

The equation of vis viva may be put into another shape which is sometimes useful. If I be the moment of inertia of the body round the instantaneous axis of the rotation by which the whole motion of the body may be represented, then

$$\Sigma mv^2 = I\omega^2.$$

Again, if y' and z' be the coordinates of any point referred to the instantaneous centre of rotation as origin,

$$\frac{dy'}{dt} = -\omega z', \quad \frac{dz'}{dt} = \omega y',$$

whence the equation of vis viva assumes the form

$$\frac{d}{dt}(I\omega^2) = 2\Sigma\left(Y\frac{dy'}{dt} + Z\frac{dz'}{dt}\right) = 2\omega\Sigma(y'Z - z'Y);$$

therefore $$\frac{1}{2\omega}\,\frac{d}{dt}(I\omega^2) = J, \tag{3}$$

where J is the moment of the applied forces round the instantaneous axis of rotation.

222. **Equation for the Moment of Momentum.**—If b and c be the coordinates of any point, fixed or movable, the moment of the applied forces, round an axis through it parallel to the axis of x, must be equal and opposite to the moment of the forces of inertia round the same; hence, calling the former moment J, we have

$$\Sigma m\left\{(y-b)\frac{d^2z}{dt^2} - (z-c)\frac{d^2y}{dt^2}\right\} = J.$$

If, as in Art. 205, we put $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$, we get, by omitting the terms which vanish,

$$\mathfrak{M}\left\{(\bar{y}-b)\frac{d^2\bar{z}}{dt^2} - (\bar{z}-c)\frac{d^2\bar{y}}{dt^2} + k^2\frac{d\omega}{dt}\right\} = J. \tag{4}$$

If we suppose the point b, c to coincide with the origin fixed in space, and to lie in the plane of the motion of the centre of inertia, this equation becomes, if we call r and χ the polar coordinates of the centre of inertia,

$$\mathfrak{M}\left\{\frac{d}{dt}\left(r^2\frac{d\chi}{dt}\right) + k^2\frac{d\omega}{dt}\right\} = J. \tag{5}$$

223. **Equations of Motion for Impulsive Forces.**—In the case of impulsive forces the equations of Art. 219 become

$$\mathfrak{M}\dot{\bar{y}} = \Sigma \underset{\cdot}{Y}, \quad \mathfrak{M}\dot{\bar{z}} = \Sigma \underset{\cdot}{Z}, \quad \mathfrak{M}k^2\dot{\theta} = \underset{\cdot}{L}. \qquad (6)$$

224. **Determination of Stress in Initial Motion.**—Stresses are determined, as we have seen, by using the dynamical equations for a free body, and introducing unknown reactions instead of the geometrical conditions. In many cases where the general equations of motion cannot be integrated, the initial stresses may be obtained by differentiating the geometrical equations twice, and introducing into the equations thus obtained the initial values of $\frac{dx}{dt}$, &c., which are supposed given, and the initial values of the coordinates. The initial values of the accelerations are then in general determined, and thence, by means of the dynamical equations, the unknown reactions.

225. **Friction.**—Friction (*see* Art. 59) is a tangential force passing through the point of contact of two rough surfaces, which tends to prevent the one from slipping on the other. If there be slipping, the friction is in an opposite direction, and takes its greatest possible value, which is in a constant ratio to the normal pressure between the surfaces. If the motion be pure rolling, just enough friction is exerted to make the motion pure rolling. The force of friction is then usually less than its maximum value, and is determined, as if it were an unknown reaction, by means of the equations of motion and the geometrical condition which expresses that the motion is pure rolling. If the value thus found for the force of friction does not exceed its maximum value, and pure rolling be consistent with the initial conditions, it will be the actual motion. When there is slipping, the friction, which is then a maximum, and therefore determined, tends to make the motion pure rolling. If pure rolling be attained, the friction at the instant pure rolling commences changes in general its value, and must be determined in the manner stated above.

If greater accuracy be required in the determination of the motion, it is necessary to take into account not only the tangential force of friction, but also what is called *the couple of rolling friction*, which is a couple having for its axis the tangent to the rough surface round which the body is rotating. Its maximum value is in a constant ratio to the normal pressure, and is generally small in amount, so that in solving questions connected with friction this couple is usually neglected. The direction in which the couple of rolling friction tends to turn the body is opposite to that in which it is actually rotating. If the body be not actually rotating, but be acted on by forces tending to make it rotate, the couple of rolling friction tends to prevent rotation round a common tangent to the two rough surfaces.

If the surfaces have a relative angular velocity about the common normal, besides the tangential force of friction, and the couple of rolling friction, there is also a couple, having the normal as its axis, called *the couple of twisting friction.* This couple likewise is usually small in amount.

It is to be observed, as already stated in Art. 59, that the maximum value of friction, when slipping actually takes place, is, in general, less than its maximum value when there is no slipping, and friction is acting against a force which tends to produce slipping.

When a surface is said to be *perfectly rough* it is understood that no slipping can take place between it and any other surface with which it is in contact. The amount of force which it is capable of exerting by means of friction is, in this case, unlimited.

226. **Impact.**—When impact occurs between two smooth bodies, a mutual impulsive force is developed in the direction of the common normal. In the first period of collision this force reduces the relative normal velocity to zero. In the case of motion parallel to a fixed plane, there are for two bodies seven unknown quantities, viz., the changes in the two components of the velocity of the centre of inertia, and in the velocity of rotation for each body, and the magnitude of the mutual impulse. There are likewise seven equations to determine these quantities, viz., the six equations of motion, and the equation which expresses that the relative normal velocity is zero.

In the second period, a new mutual impulsive force is developed, whose impulse bears a constant ratio to that of the former, and can therefore be found. The changes of velocity which it produces can then be determined.

If the bodies which collide be perfectly elastic, the impulse developed during the period of restitution, or second period, is equal to that developed during the first or period of compression. What is here stated is merely a generalization of the theory given in Article 76.

227. **Impulsive Friction.**—When two rough surfaces collide, the investigation of what takes place is, in general, somewhat complicated. We must regard R and F, the impulses of the normal reaction and friction, as variable quantities, connected at each instant of the impact, by linear equations with the coexisting values of the velocities of rotation and of translation of the centres of inertia. The laws which regulate the impulse of friction may then be stated as follows:—

(1) The direction of the elementary impulse dF due to friction is opposite to that of the slipping of the point of contact, if there be slipping, and if there be no slipping, is such as to prevent slipping.

(2) The magnitude of dF is, if possible, just sufficient to prevent slipping, but dF can never exceed μdR, and when slipping takes place $dF = \mu dR$.

The equations of motion for impulses (Art. 223) show that the relative normal and tangential velocities of the point of contact are, at each instant, of the form $AR + BF + C$, where A, B, and C are constant during the impact.

The value of R is at first zero; when it becomes R_1 (at the end of the first period of the impact), the relative normal velocity is equal to zero; and the maximum value of R, which it assumes at the end of the whole impact, is equal to $(1 + e) R_1$.

These principles afford a sufficient number of equations to determine the motion; and, in the case of motion parallel to a fixed plane, the equations are always soluble.

If the bodies which collide are *perfectly rough*, the relative tangential velocity, or velocity of slipping, is always zero; and when $R = R_1$, the relative normal velocity is likewise zero.

Hence we have two equations to determine R_1 and the corresponding value of F. At the end of the impact $R = (1 + e) R_1$; and the relative tangential velocity being still zero, the corresponding value of F can be determined.

If the bodies *slip on each other in the same direction during the whole of the impact*, dF is always equal to μdR; hence $F = \mu R$ throughout. R_1 is then determined from the equation expressing that the relative normal velocity is zero; and the final values of R and F, which determine the motion after the impact, are

$$(1 + e) R_1, \quad \text{and} \quad \mu (1 + e) R_1.$$

For a discussion of the problem in more complicated cases the reader is referred to Routh, *Rigid Dynamics*, Art. 187–198.

If a *sphere* impinges against a *fixed* surface, or *two spheres* collide with each other, the relative tangential velocity V is independent of the normal reaction, and the relative normal velocity is independent of the friction. In this case, if V become zero it must remain zero, as friction cannot initiate a relative tangential velocity in its own line of direction. Hence V must be either zero at the end of the impact, or in the same direction as at the beginning. Moreover, the value of R_1 is independent of F. The problem is, therefore, reducible to one of the two cases treated above.

EXAMPLES.

1. A homogeneous cylinder, having its axis horizontal, rolls without slipping down a rough inclined plane; neglecting the couple of rolling friction, determine the amount of friction brought into play (*see* Ex. 1, p. 136).

The equations of motion are

$$M\frac{d^2y}{dt^2} = Mg \sin i - F, \quad Mk^2\frac{d^2\theta}{dt^2} = aF;$$

the axis of y being a line in the inclined plane at right angles to its intersection with the horizon. Also, $a\,d\theta = dy$; whence

$$F = Mg \sin i \frac{k^2}{a^2 + k^2};$$

but $$k^2 = \frac{a^2}{2}, \quad \text{and therefore} \quad F = \frac{Mg \sin i}{3}.$$

2. If a sphere be substituted for a cylinder in the last example, determine the amount of friction brought into play.

Ans. $F = \frac{2}{7} Mg \sin i.$

3. A plane area is made to rotate with an angular velocity ω round a fixed axis in its own plane by the expenditure of a given amount of work. When rotating it strikes a sphere of mass m, at a distance a from the fixed axis, whose velocity at the instant of impact is zero. Determine the moment of inertia of the plane area round the fixed axis in order that the velocity imparted to the sphere should be a maximum.

If R be the impulse on the sphere in the first period of impact, and v its velocity at the end of this period,

$$mv = R, \quad I\dot{\theta} = -aR, \quad a(\omega + \dot{\theta}) = v,$$

whence
$$R = \frac{maI\omega}{I + ma^2}.$$

The whole impulse given to the sphere is $(1 + e)R$. Hence R is to be a maximum; but $I\omega^2 =$ given constant; therefore $\frac{\sqrt{I}}{I + ma^2} =$ maximum, and therefore $I = ma^2$.

4. An inextensible string, whose mass is negligible, passes over the line of intersection of two smooth inclined planes. Each end of the string passes under and round a smooth circular homogeneous cylinder, to which it is attached, and which rests on one of the inclined planes. The line of intersection of the inclined planes is parallel to the axes of the cylinders, and perpendicular to a vertical plane containing their centres of inertia and the string. Determine the tension of the string.

As in Ex. 6, Art. 218, the portion of the string wrapped round one of the cylinders may be regarded as in equilibrium under the action of the tensions at its extremities, and of the pressure produced by the cylinder. Hence all the forces exerted by the string on the cylinder are equivalent to the tension T acting at the point of contact of the cylinder with the inclined plane.

If s and s' be the distances at any time of the points of contact of the cylinders and inclined planes, from the point of intersection of the latter with the vertical plane perpendicular to them; θ and θ' the angles through which the cylinders have turned from their initial positions; a and a' their radii; m and m' their masses, and i and i' the inclinations of the inclined planes to the horizon, the equations of motion are

$$m\frac{d^2s}{dt^2} = mg\sin i - T, \qquad m\frac{a^2}{2}\frac{d^2\theta}{dt^2} = Ta,$$

$$m'\frac{d^2s'}{dt^2} = m'g\sin i' - T, \qquad m'\frac{a'^2}{2}\frac{d^2\theta'}{dt^2} = Ta'.$$

If σ be the distance the string has slipped at any time along the inclined planes, and b and b' the initial values of s and s', we have, since the string is inextensible,

$$s = b + a\theta + \sigma, \quad s' = b' + a'\theta' - \sigma, \quad \text{and therefore } s + s' = b + b' + a\theta + a'\theta'.$$

Differentiating twice we obtain, by means of the equations of motion,

$$T = \tfrac{1}{3}\frac{mm'}{m + m'}g(\sin i + \sin i').$$

The motion can then be completely determined.

5. The base of a smooth homogeneous circular semi-cylinder rests on a horizontal plane. A heavy particle is placed at a point on the surface of the semi-cylinder, situated in a vertical plane containing its centre of inertia and perpendicular to its axis. Show that the particle will describe an ellipse.

Let the axis of x be the intersection of the vertical plane, in which the particle moves, with the horizontal plane on which the semi-cylinder rests; the axis of y being vertical. Let x, y be the coordinates of the particle, m its mass, x' the coordinate of the centre of inertia of the semi-cylinder, m' its mass, and a its radius.

Considering the whole system as one body, we have (Art. 203),

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0.$$

Hence, since the system starts from rest, $mx + m'x'$ is constant, or the projection on the horizontal plane of the centre of inertia of the whole system remains fixed in space. Taking this point for origin, we have $mx + m'x' = 0$.

Again, since the semi-cylinder is homogeneous, we have, from the geometrical conditions,

$$(x - x')^2 + y^2 = a^2.$$

Substituting for x' we obtain

$$(m + m')^2 x^2 + m'^2 y^2 = m'^2 a^2.$$

6. A homogeneous bar falling freely without rotation impinges upon a smooth horizontal plane; find the velocity of rotation immediately after impact.

If R be the impulse exerted by the plane on the bar in the first period of impact, V the velocity of any point of the bar just before the impact, m its mass, and a half its length; since the direction of R is vertical, the equations for determining the angular velocity ω are,

$$m\dot{\bar{z}} = R, \quad mk^2\omega = aR\cos\alpha, \quad V + \dot{\bar{z}} + a\omega\cos\alpha = 0,$$

where α is the angle which the bar makes with the horizontal plane; whence,

as $k^2 = \dfrac{a^2}{3}$, we have $\omega = \dfrac{-3\cos\alpha}{1 + 3\cos^2\alpha} \cdot \dfrac{V}{a}$,

7. In what direction must an impulse be applied to a sphere in order that its initial motion may be one of rotation round a given tangent?

The direction of the initial motion of the centre of inertia of the sphere is in this case given. Hence the direction of the impulse is a line parallel to this, lying in the plane, which passes through the centre, at right angles to the given tangent, and distant from the centre by $\frac{2}{5}$ radius.

8. A beam placed in a smooth horizontal plane is turning with a given velocity round a pivot which passes through a given point. The pivot breaks; determine the subsequent motion.

If b be the distance of the centre of inertia of the beam from the pivot, this point of the beam continues to move with a constant velocity $b\omega$ in the straight line which is at right angles to the beam at the moment when the pivot breaks, and the beam rotates with a constant angular velocity ω round a vertical axis through its centre of inertia.

9. If a hoop rolls down a rough inclined plane without sliding, show that $\tan i < 2\mu$; the initial position of the hoop being in a vertical plane at right angles to the intersection of the inclined plane with the horizon, and the couple of rolling friction being neglected.

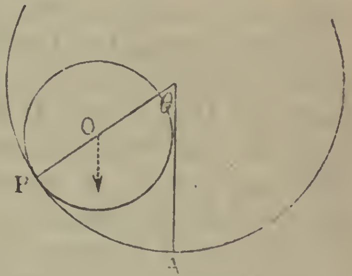
Take the initial position of the centre of the hoop for origin, and the intersection of the inclined plane with a vertical plane at right angles thereto as axis of y, its positive direction being downwards. Let the positive direction of rotation be from the upper side of the inclined plane towards y positive. Then, y being the coordinate of the centre of the hoop, m its mass, a its radius, and F the friction brought into play, the equations of motion are

$$ma^2 \frac{d\omega}{dt} = Fa, \quad m\frac{d^2y}{dt^2} = mg \sin i - F;$$

but the motion being pure rolling, $a\omega = \dfrac{dy}{dt}$; hence, eliminating, we obtain

$$F = \frac{mg \sin i}{2}; \text{ but } F < \mu mg \cos i; \text{ therefore } \tan i < 2\mu.$$

10. A homogeneous circular disk, whose radius is a, rolls inside a rough vertical circle whose radius is b; the motion is pure rolling under the action of gravity; show that the rolling forward and backward of the disk is isochronous with the oscillations of a simple pendulum whose length is $\frac{3}{2}(b-a)$, the couple of rolling friction being neglected.



We have, I being the moment of inertia of the disk round an axis through P, ω the angular velocity, and θ the angle between CA and CP, $\dfrac{1}{\omega}\dfrac{d}{dt}(I\omega^2) = -2\mathfrak{M}ga \sin\theta$ (θ being reckoned from the vertical line CA, where C is the centre of the vertical circle, and ω being the angular velocity of the disk rolling up). As the instantaneous centres of rotation are, in this case, points on the circumference of the disk, I remains constant throughout the motion; therefore $I\dfrac{d\omega}{dt} = -\mathfrak{M}ga \sin\theta$; but I (*Integral Calculus*, Chap. X.) $= \frac{3}{2}\mathfrak{M}a^2$, and $a\omega = (b-a)\dfrac{d\theta}{dt}$, since either represents the velocity of O. Hence

$$\tfrac{3}{2}(b-a)\frac{d^2\theta}{dt^2} = -g \sin\theta; \ \therefore \ \&c.$$

The student will observe that the force of friction at P does not enter this equation. It is also to be observed, that under the circumstances supposed the disk would go on rolling forwards and backwards for ever.

11. A body is moving parallel to a fixed plane, under the action of forces which are in equilibrium: show that the locus of the instantaneous centre of rotation in the body is a circle, having the centre of inertia for centre, and a radius $\dfrac{v}{\omega}$, where v is the velocity of the centre of inertia, and ω the angular velocity.

12. The locus of the instantaneous centre of rotation in space, under the circumstances of Ex. 11, is a straight line parallel to the path of the centre of inertia, and at a distance from it equal to $\frac{v}{\omega}$.

13. If a body move parallel to a fixed plane, and be acted on by a constant couple, lying in the plane; show that the locus of the instantaneous centre of rotation in space is an equilateral hyperbola.

228. **Tendency of a Rod to Break.**—When a body is under the influence of any forces, acting either with finite magnitude at isolated points, or continuously throughout its entire mass, it experiences pressures or tensions, which tend to alter the relative positions of its molecules. This tendency is resisted by the mutual action of the molecules. Under such circumstances the body is said to be in a state of *stress*.

If we consider a small rectangular parallelepiped in the body, the stresses acting on one of its faces may be resolved into three forces at right angles to each other—one normal, and two parallel to the face under consideration.

To ascertain the tendency of a body to undergo a rupture in any part, we must consider the stresses to which it is subjected in that part. If the mutual cohesion of the molecules is unable to resist these stresses the body must give way. The question is, in general, one of great complication, and for its full discussion the reader is referred to treatises on *Elasticity* and *Strength of Materials*.

If the body under consideration be a *rod*, that is, if two of its dimensions are at each point very small, the question becomes much simplified.

The axis of the rod may be a straight line, or may form a curve of any kind.

We shall suppose that this curve is not closed, that it lies in one plane P, and that the rod is in equilibrium under the action of forces in this plane. If we consider a section at right angles to the axis of the rod, at any point A of its length, the action of the molecules at one side of this section on those at the other must equilibrate all the forces acting on the rod at the latter side. These may be reduced to a force F, passing through A, and a couple G, round an axis a, at right angles to the plane P. This force and couple therefore are equivalent to the stresses acting on the rod through the section containing A.

That the tendency of the rod to break results chiefly from the couple may be shown as follows:—

The stresses in the plane of the section cannot give any couple round the axis a, since a either meets them or is parallel to them. Hence the couple G must produce stresses, parallel to the axis of the rod at the point A, whose moment round A is equal to G. If N be the value per unit of area of the greatest of these stresses, and a be the distance from A of the most remote point of the section, whose area may be denoted by S; the moment round A of the stresses parallel to the axis must be less than NSa. Hence, if we assume $G = Fp$, we have

$$NSa > Fp, \text{ and therefore } N > \frac{F}{S}\frac{p}{a}.$$

If we now seek for the stress per unit of area caused by the force F, we have $N' = \dfrac{F}{S}$.

If we compare N and N' we see that, if a is very small compared with p, N' is unimportant compared with N. Hence, in general, the tendency of the rod to break at any point A depends simply on N, or, on G, the moment round A of the forces acting on the rod at all the points of its length on one side of A.

We have hitherto supposed the rod to be in equilibrium. If it be in motion, we can, by D'Alembert's Principle, consider it as in equilibrium under the action of the applied forces and the forces of inertia, and the question of stress, or tendency to break at any point, becomes the same as before, except that we must now add the forces of inertia to the other forces acting on the rod.

To find the moment round a point b, c of the rod, of the forces of inertia of that portion of the rod which is on one side of b, c.

Let m' be the mass of this portion of the rod, k' its radius of gyration round the centre of inertia of the whole rod, k'' its radius of gyration round its own centre of inertia; y', z' the coordinates of its centre of inertia referred to the fixed origin, and η', ζ' its coordinates referred to parallel axes through the centre of inertia of the whole rod, then the moment of the forces of inertia of m' round the point b, c is, by Art. 222,

$$-m'\left\{(y'-b)\frac{d^2z'}{dt^2}-(z'-c)\frac{d^2y'}{dt^2}+k''^2\frac{d\omega}{dt}\right\}.$$

Also, since

$$y' = \bar{y} + \eta', \quad z' = \bar{z} + \zeta', \quad \frac{d\eta'}{dt} = -\zeta'\omega, \quad \frac{d\zeta'}{dt} = \eta'\omega,$$

we have

$$\frac{d^2y'}{dt^2} = \frac{d^2\bar{y}}{dt^2} - \zeta'\frac{d\omega}{dt} - \eta'\omega^2,$$

$$\frac{d^2z'}{dt^2} = \frac{d^2\bar{z}}{dt^2} + \eta'\frac{d\omega}{dt} - \zeta'\omega^2;$$

whence, substituting and remembering that

$$\frac{d^2\bar{y}}{dt^2} = \frac{\Sigma Y}{m}, \quad \frac{d^2\bar{z}}{dt^2} = \frac{\Sigma Z}{m},$$

and that $$k'^2 = k''^2 + \eta'^2 + \zeta'^2,$$

we obtain for the required moment

$$-m'\left\{(y'-b)\frac{\Sigma Z}{m} - (z'-c)\frac{\Sigma Y}{m}\right.$$
$$\left. + \frac{d\omega}{dt}\left[k'^2 + (\bar{y}-b)\eta' + (\bar{z}-c)\zeta'\right] - \omega^2\left[(y'-b)\zeta' - (z'-c)\eta'\right]\right\}.$$

If the rod be straight, since $\eta' = y' - \bar{y}$, $\zeta' = z' - \bar{z}$, and b, c, y', z', $\bar{y}, \bar{z}$, are each points on the rod, the coefficient of ω^2 vanishes, and the moment of the forces of inertia of m' round b, c becomes

$$-m'\left\{(y'-b)\frac{\Sigma Z}{m} - (z'-c)\frac{\Sigma Y}{m} + \frac{d\omega}{dt}\left[k'^2 + (\bar{y}-b)\eta' + (\bar{z}-c)\zeta'\right]\right\}.$$

To find the couple G which measures the tendency of a rod to break at any point of its length.

Adopting the same notation as above, if L' be the moment round this point of the forces applied to the portion of the rod at one side of the point, we have

$$G = L' - m'\left\{(y'-b)\frac{\Sigma Z}{m} - (z'-c)\frac{\Sigma Y}{m}\right.$$
$$\left. + \frac{d\omega}{dt}\left[k'^2 + (\bar{y}-b)\eta' + (\bar{z}-c)\zeta'\right] - \omega^2\left[(y'-b)\zeta' - (z'-c)\eta'\right]\right\}. \quad (7)$$

If Λ' be the moment round b, c of the acceleration of y', z', G may be put into another form, which follows immediately from the first expression for the moment of the forces of inertia, viz.,

$$G = L' - m'\left(\Lambda' + k''^2 \frac{d\omega}{dt}\right). \qquad (8)$$

If the rod be free, equation (7) is usually most convenient; whilst, on the other hand, if it be constrained to move in a particular way, equation (8) may sometimes be used with advantage.

The first expression for G may, if we please, be written

$$G = L' - m'\left\{(y' - b)\frac{d^2\bar{z}}{dt^2} - (z' - c)\frac{d^2\bar{y}}{dt^2}\right.$$

$$\left. + \frac{d\omega}{dt}\left[k'^2 + (\bar{y} - b)\eta' + (\bar{z} - c)\zeta'\right] - \omega^2\left[(y' - b)\zeta' - (z' - c)\eta'\right]\right\}. \qquad (9)$$

EXAMPLES.

1. A uniform rod is turning in a vertical plane round a horizontal pivot A, at one of its extremities. Find the tendency to break at any point P.

Let the length of the rod $= 2a$, and let ξ be the distance of the point P from the pivot; then $m' = m\dfrac{2a - \xi}{2a}$. If C be the centre of inertia of m', $PC = \dfrac{2a - \xi}{2}$, wherefore, if θ be the angle which the rod makes with the horizontal line

$$L' = \frac{2a - \xi}{2} m'g \cos\theta.$$

Moreover, since C is moving in a circle round A as centre, its acceleration has two components—one at right angles to PC, which is

$$\left(a + \frac{\xi}{2}\right)\frac{d^2\theta}{dt^2},$$

and the other along PC. The latter gives no moment round P; hence

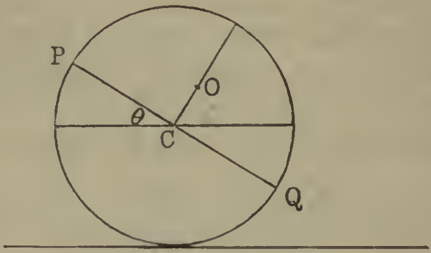
$$\Lambda' = \frac{2a + \xi}{2}\,\frac{2a - \xi}{2}\,\frac{d^2\theta}{dt^2},$$

and
$$G = \frac{2a - \xi}{2} m'g \cos\theta - m'\left\{\frac{4a^2 - \xi^2}{4}\,\frac{d^2\theta}{dt^2} + k''^2 \frac{d^2\theta}{dt^2}\right\};$$

but
$$\frac{4}{3}ma^2\frac{d^2\theta}{dt^2} = mga\cos\theta, \quad \text{and} \quad k''^2 = \frac{(2a-\xi)^2}{12};$$

whence
$$G = -\,mg\,\frac{(2a-\xi)^2}{16a^2}\,\xi\cos\theta.$$

2. A cracked hoop rolls on a perfectly rough horizontal plane. Determine the inclination to the horizon of the line joining the crack to the opposite point when the tendency to break at this point is the greatest possible.



In this case the centre of inertia of the hoop moves in a straight line with a constant velocity. Hence its acceleration is zero, also

$$\frac{d\omega}{dt} = 0, \quad L' = \frac{m}{2}g\left(a\cos\theta + \frac{2a}{\pi}\sin\theta\right), \text{ since } CO = \frac{2a}{\pi},$$

O being the centre of inertia of the semi-hoop comprised between the crack Q and the opposite point.

Taking PC and CO for the directions of the axes,

$$y' - b = a, \quad z' - c = \frac{2a}{\pi}, \quad \eta' = 0, \quad \zeta' = \frac{2a}{\pi}.$$

Hence

$$G = L' + m'\omega^2\left\{(y'-b)\,\zeta' - (z'-c)\,\eta'\right\} = \frac{m}{2}\left\{g\left(a\cos\theta + \frac{2a}{\pi}\sin\theta\right) + \frac{2a^2}{\pi}\,\omega^2\right\};$$

consequently G is greatest when

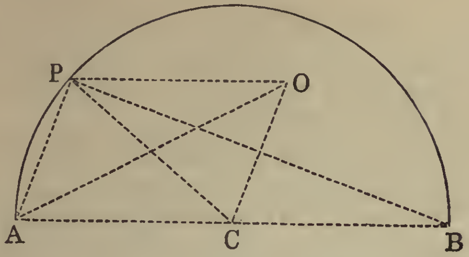
$$\cos\theta + \frac{2}{\pi}\sin\theta \text{ is a maximum, } i.e. \text{ when } \tan\theta = \frac{2}{\pi}.$$

3. A semicircular wire, of radius a, lying on a smooth horizontal table, turns round one extremity A, with a constant angular velocity ω. Find the tendency to break at any point P.

Let O be the centre of inertia of the arc PB, and let $PCA = \phi$. Join AO, AP, and PO; then, since the angular velocity is constant, the acceleration of O is $\omega^2 . AO$. Consequently $\frac{\Lambda'}{\omega^2}$ is double the area of the triangle APO; but, since AP and CO are parallel, the triangle APO is equal to the triangle ACP, or equal to $a^2\sin\phi$.

Hence
$$\Lambda' = a^2\omega^2\sin\phi, \quad \text{and} \quad G = m\frac{\pi-\phi}{\pi}\,a^2\omega^2\sin\phi.$$

Accordingly the tendency to break is a maximum at the point determined by the equation $\tan\phi = \pi - \phi$.



This example and the two preceding are taken from Routh, *Rigid Dynamics.*

4. A uniform sphere, resting on a rough horizontal plane, is set in motion by an impulse applied in a vertical plane passing through its centre. Show that, when sliding ceases, the rolling motion will be direct, stationary, or retrograde, according as the direction of the impulse intersects the vertical diameter above, at, or below the point of contact with the plane. Impulsive friction and the couple of rolling friction are neglected (Mr. Townsend).

Let v be the velocity, at any time, of the centre of the sphere parallel to the intersection of the horizontal plane with the vertical plane containing the impulse; the direction of the latter making an acute angle with the positive direction of v. Let ω be the angular velocity of the sphere, counted from the vertical towards the direction of v positive: then V and Ω, the initial values of v and ω, are determined by the equations

$$mV = \underset{\cdot}{Y}, \quad mk^2\Omega = \underset{\cdot}{Y}b,$$

where $\underset{\cdot}{Y}$ is the horizontal component of the impulse, and b the distance from the centre, at which its line of direction intersects the vertical diameter of the sphere. Eliminating $\underset{\cdot}{Y}$, we obtain

$$\Omega = \frac{bV}{k^2}.$$

For the subsequent motion, if F be the force of friction, we have the equations

$$m\frac{dv}{dt} = F, \quad mk^2\frac{d\omega}{dt} = -Fa;$$

whence $$a\frac{dv}{dt} + k^2\frac{d\omega}{dt} = 0,$$

Integrating, we obtain

$$av + k^2\omega = \text{constant} = aV + k^2\Omega = (a+b)V.$$

When sliding ceases, $v = a\omega$, Substituting for v in the preceding equation, we have

$$\omega = \frac{(a+b)V}{a^2+k^2}$$

Hence, since V is necessarily positive, ω, when sliding ceases, is positive, zero, or negative, according as

$$a + b > 0, \quad a + b = 0, \quad \text{or} \quad a + b < 0.$$

The first condition holds good, if b is either positive, or negative and less than a in absolute magnitude; the second, if $b = -a$; the third, if b is negative and greater than a in absolute magnitude.

The results of this example may be extended to other solids of revolution.

5. The extremities of a heavy rod are attached by cords of equal length to a horizontal beam. One cord is cut; find the initial tension on the other, and determine the angle at which the cords should intersect the beam, in order that the tension should be the greatest possible.

Let ϕ be the angle which the cord makes with the horizontal beam at any time, and θ the angle which the rod makes with its initial position; then, if T be the tension of the cord, l its length, $2a$ the length of the rod, and y and z the coordinates of its centre of inertia,

$$m \frac{a^2}{3} \frac{d^2\theta}{dt^2} = a \sin(\phi - \theta) T,$$

$$m \frac{d^2y}{dt^2} = -T \cos\phi, \quad m \frac{d^2z}{dt^2} = mg - T \sin\phi,$$

the origin being taken at the point of the fixed beam to which the uncut cord is attached. Also

$$y = l \cos\phi + a \cos\theta, \quad z = l \sin\phi + a \sin\theta;$$

whence, differentiating twice, and remembering that initially

$$\frac{d\phi}{dt} = 0, \quad \frac{d\theta}{dt} = 0, \quad \phi = \alpha, \quad \theta = 0,$$

we obtain for the determination of the initial tension

$$\frac{ma^2}{3} \frac{d^2\theta}{dt^2} = aT \sin\alpha, \quad -ml \sin\alpha \frac{d^2\phi}{dt^2} = -T \cos\alpha,$$

$$ml \cos\alpha \frac{d^2\phi}{dt^2} + ma \frac{d^2\theta}{dt^2} = mg - T \sin\alpha;$$

whence
$$T = mg \frac{\sin\alpha}{1 + 3\sin^2\alpha}.$$

This is a maximum when

$$\alpha = \sin^{-1} \frac{1}{\sqrt{3}}.$$

6. A uniform bar, of length $2a$, is suspended from a fixed, parallel and equal, horizontal bar, by strings each having a length $2a$ joining the adjacent extremi-

ties of the bars. An angular velocity ω is imparted to the suspended bar round a vertical axis through its centre of inertia. Determine the vertical height through which its centre of inertia will rise.

As each extremity of the bar moves on the surface of a sphere to which the attached string is radius, the tensions of the strings do not appear in the equation of vis viva; hence $h = \dfrac{a^2\omega^2}{6g}$.

7. Determine in the last example the initial tensions of the strings, and their tensions when the bar is at its greatest height.

If θ be the angle one of the strings makes with a vertical line at any time, ψ the angle the bar makes with a horizontal line parallel to the fixed bar, and T the tension of one string; then

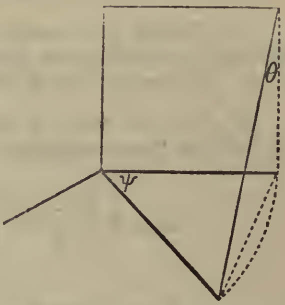
$$m\frac{a^2}{3}\frac{d^2\psi}{dt^2} = -2aT\sin\theta\cos\tfrac{1}{2}\psi,$$

$$m\frac{d^2z}{dt^2} = 2T\cos\theta - mg;$$

also, from the geometrical conditions,

$$\theta = \tfrac{1}{2}\psi, \text{ as } 2a\sin\theta = 2a\sin\tfrac{1}{2}\psi,$$

$$z = 2a(1-\cos\theta).$$



Substituting $\frac{1}{2}\psi$ for θ in the last equation, differentiating twice, and observing that initially $\psi = 0$, and $\dfrac{d\psi}{dt} = \omega$, we get for the initial tension of one string $T = \dfrac{mg}{2} + \frac{1}{4}ma\omega^2$.

To get the tension when the bar is at its highest position, make $\dfrac{d\psi}{dt} = 0$, $\cos\frac{1}{2}\psi = \dfrac{2a-h}{2a}$, where h has the value in Ex. 6; then

$$T = mg\frac{4a^3}{(2a-h)\{4a^2 + 3h(4a-h)\}} = mg\frac{288g^3}{(12g - a\omega^2)\{48g^2 + \omega^2(24ag - a^2\omega^2)\}}.$$

8. A circular plate rolls down the inner circumference of a rough circle under the action of gravity. The plane of the plate coincides with that of the rough circle, which is vertical. Determine the amount of friction brought into play if the plate start from rest, the motion being pure rolling, and the couple of rolling friction being neglected. (See Example 9, Art. 227.)

The equations of motion are

$$\tfrac{1}{2}Ma^2\frac{d^2\psi}{dt^2} = Fa, \quad M(b-a)\frac{d^2\theta}{dt^2} = -Mg\sin\theta + F$$

together with the equation of condition

$$(b-a)\frac{d\theta}{dt} = -a\frac{d\psi}{dt};$$

whence $F = \frac{1}{3}Mg\sin\theta$.

9. Show that the plate in the last example will ascend to the same height as that from which it started, and that the motion will go on for ever.

By vis viva $\Sigma mv^2 = 2Mg(z - z_0)$, this will vanish when $z = z_0$; therefore, &c.

10. Determine the velocity of rotation of the plate at any time.

This is most easily done by means of the equation $\frac{1}{\omega}\frac{d}{dt}(I\omega^2) = 2J$ (Art. 221), whence

$$\omega^2 = \tfrac{4}{3}\,\frac{(b-a)g}{a^2}\,(\cos\theta - \cos\theta_0).$$

This result can also be obtained from the equations of Ex. 8.

11. Taking into account the couple of rolling friction, and supposing the motion to be still pure rolling, determine—(1) the angular velocity; (2) the vis viva; (3) the amount of friction brought into play.

If f be the ratio of the couple of rolling friction to the normal pressure (f being a linear magnitude),

$$\omega^2 = \tfrac{4}{3}\,\frac{(b-a)g}{a^2}\left\{(\cos\theta - \cos\theta_0) - \frac{f}{a}(\sin\theta_0 - \sin\theta)\right\},$$

$$\Sigma mv^2 = 2M(b-a)g\left\{(\cos\theta - \cos\theta_0) - \frac{f}{a}(\sin\theta_0 - \sin\theta)\right\},$$

$$F = \tfrac{1}{3}Mg\sin\theta + \tfrac{2}{3}\,\frac{f}{a}\,Mg\cos\theta.$$

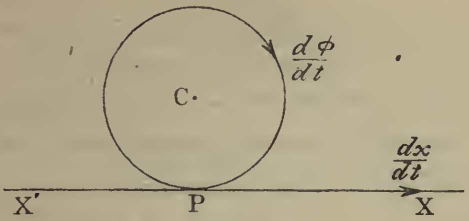
12. Determine, under the circumstances supposed in Ex. 11, the height h to which the disk will ascend.
If h_0 be the height of the initial position of the centre of the disk above its lowest position, and $r = b - a$, h is given by the equation,

$$h = h_0 - \frac{f}{a}\left\{\sqrt{(2rh_0 - h_0^2)} + \sqrt{(2rh - h^2)}\right\}.$$

It follows from this that h is less than h_0, and that the disk will finally come to rest.

13. A circular plate is projected along a rough horizontal plane, with an initial velocity v of translation, and an angular velocity ω, round an axis through its centre, at right angles to its plane. Determine the motion.

Let $\frac{d\phi}{dt}$ be the angular velocity at any time, and let x, the horizontal coordinate of the centre, be measured in the direction of v, as in the figure. The rotation $\frac{d\phi}{dt}$ round C is equivalent to $\frac{d\phi}{dt}$ round P, together with the velocity of translation, $-a\frac{d\phi}{dt}$ along PX. Hence the whole velocity of translation of P is $\frac{dx}{dt}-a\frac{d\phi}{dt}$, where a is the radius of plate.



Different phenomena present themselves according to the initial values of v and ω.

Neglecting the couple of rolling friction—

(1) ω positive, and $v > a\omega$.

Since $v-a\omega$ is positive, P begins to slip along PX; therefore $F=\mu Mg$, and the equations of motion are

$$M\frac{d^2x}{dt^2}=-\mu Mg,\quad \tfrac{1}{2}Ma^2\frac{d^2\phi}{dt^2}=\mu Mga\,;$$

hence
$$\frac{dx}{dt}-a\frac{d\phi}{dt}=0,\quad \text{when}\quad t=\frac{v-a\omega}{3\mu g}.$$

The motion then becomes pure rolling, and we have the equations

$$M\frac{d^2x}{dt^2}=-F,\quad \tfrac{1}{2}Ma^2\frac{d^2\phi}{dt^2}=Fa,$$

where F is the amount of friction brought into play.

Also
$$\frac{dx}{dt}=a\frac{d\phi}{dt},\quad \text{whence } F=0,$$

$$a\frac{d\phi}{dt}=\frac{dx}{dt}=\text{constant}=\tfrac{2}{3}v+\tfrac{1}{3}a\omega,$$

and the disk will roll on with a constant velocity of rotation round the instantaneous axis.

(2) ω positive as before, $v < a\omega$.

Since $v-a\omega$ is negative, P commences by slipping back towards X'; the equations of motion are then,

$$M\frac{d^2x}{dt^2}=\mu Mg,\quad \tfrac{1}{2}Ma^2\frac{d^2\varphi}{dt^2}=-\mu Mga.$$

When $t = \dfrac{a\omega - v}{3\mu g}$ pure rolling commences, and, as before,

$$\frac{dx}{dt} = \tfrac{2}{3}v + \tfrac{1}{3}a\omega = a\frac{d\phi}{dt}.$$

The initial velocity of translation of the centre is, in this case, increased.

(3) Initial angular velocity negative and equal to $-\omega$.

The initial velocity of P is then $v + a\omega$. Slipping commences and continues till $t = \dfrac{v + a\omega}{3\mu g}$, after which pure rolling begins, and continues for ever; the velocity of the centre being given by the equation

$$\frac{dx}{dt} = a\frac{d\phi}{dt} = \tfrac{2}{3}v - \tfrac{1}{3}a\omega.$$

If $a\omega > 2v$, $\dfrac{dx}{dt}$ and $\dfrac{d\phi}{dt}$ will both be negative, that is, the motion of translation of the centre will be in the direction opposite to that originally imparted, and the rotation will be in the same direction as the initial rotation.

If we take into account the couple of rolling friction, the equations of motion become, for (1),

$$M\frac{d^2x}{dt^2} = -\mu Mg, \quad \tfrac{1}{2}Ma^2\frac{d^2\phi}{dt^2} = \mu Mga - fMg.$$

Pure rolling commences when $t = \dfrac{v - a\omega}{\left(3\mu - 2\dfrac{f}{a}\right)g}$:

at this instant
$$\frac{dx}{dt} = \frac{2v\left(\mu - \dfrac{f}{a}\right) + \mu a\omega}{3\mu - 2\dfrac{f}{a}} = \left(\frac{dx}{dt}\right)_0.$$

After this the equations of motion become

$$M\frac{d^2x}{dt^2} = -F, \quad \tfrac{1}{2}Ma^2\frac{d^2\phi}{dt^2} = Fa - fMg,$$

along with $\dfrac{dx}{dt} = a\dfrac{d\phi}{dt}$; whence $F = \tfrac{2}{3}\dfrac{f}{a}Mg$.

This expression shows that the friction brought into play varies inversely as the radius of the plate, provided its mass be constant. The plate will come to rest at a time

$$t' = \frac{3}{2}\,\frac{\left(\dfrac{dx}{dt}\right)_0}{\dfrac{f}{a}g},$$

where t' is counted from the instant when pure rolling begins.

In order that pure rolling should continue, as assumed above, it is necessary that $\mu > \frac{2}{3}\frac{f}{a}$.

The student will have no difficulty in investigating cases (2) and (3) when the couple of rolling friction is taken into account.

14. A sphere, rotating with an angular velocity ω round a horizontal axis at right angles to the plane of the trajectory of its centre, impinges on a perfectly rough horizontal plane: find the motion immediately after impact.

Suppose the sphere is moving from left to right before impact with velocity v, whose direction makes an angle i with the plane of the horizon. Let ω' be the angular velocity in the direction of the motion of the hands of a watch, and v' the horizontal velocity of the centre from left to right at the instant after impact. F being the impulse arising from friction, the equations of motion are,

$$Mv' = Mv \cos i + F,$$

$$\tfrac{2}{5}Ma^2\omega' = \tfrac{2}{5}Ma^2\omega - aF.$$

The geometrical condition for no slipping is

$$v' - a\omega' = 0;$$

whence

$$\frac{F}{M} = -\tfrac{2}{7}(v \cos i - a\omega),$$

$$v' = a\omega' = \tfrac{5}{7}v \cos i + \tfrac{2}{7}a\omega.$$

If $v \cos i = a\omega$, no impulsive friction is called into play. If $v \cos i > a\omega$, the horizontal velocity of the centre of the sphere is diminished, and the sphere rebounds at a greater angle than if there were no friction. If $v \cos i < a\omega$ the horizontal velocity of the sphere is increased, and the sphere rebounds at a smaller angle than if there were no friction. In this case friction accelerates the horizontal velocity of the centre of the sphere.

If ω is opposite in direction to the motion of the hands of a watch,

$$v' = \tfrac{5}{7}v \cos i - \tfrac{2}{7}a\omega.$$

The velocity of the centre of the sphere along the horizontal line is diminished, and the sphere will rebound at a greater angle than if there were no friction. If $5v \cos i = 2a\omega$ the sphere will rebound vertically. If $2a\omega > 5v \cos i$ the sphere will hop back. This explains the effect of slow under-cut in tennis. The numerical factors for a tennis ball may of course be different from those given above.

15. If the plane in the last example be imperfectly rough, so that the impulsive friction is not sufficient to destroy the whole tangential velocity of the point of contact of the sphere with the plane, determine the motion.

The equations are—(1) If $v \cos i > a\omega$,

$$Mv' = Mv \cos i - \mu(1 + e)\,Mv \sin i,$$

$$\tfrac{2}{5}Ma^2\omega' = \tfrac{2}{5}Ma^2\omega + \mu(1 + e)\,Mva \sin i.$$

(2) If $v \cos i < a\omega$,

$$Mv' = Mv \cos i + \mu(1+e) Mv \sin i,$$

$$\tfrac{2}{5} Ma^2\omega' = \tfrac{2}{5} Ma^2\omega - \mu(1+e) Mva \sin i.$$

(3) If ω be in the opposite direction to the motion of the hands of a watch,

$$Mv' = Mv \cos i - \mu(1+e) Mv \sin i,$$

$$\tfrac{2}{5} Ma^2\omega' = -\tfrac{2}{5} Ma^2\omega + \mu(1+e) Mva \sin i.$$

16. A sphere is projected down a rough inclined plane, along the line of intersection of the plane with the vertical plane at right angles thereto. The sphere has an initial velocity of rotation round a horizontal axis parallel to the inclined plane; determine the motion—(1) neglecting the couple of rolling friction; (2) taking into account the couple of rolling friction.

Let the line of projection be the axis of x, and let x positive be measured to the right, and $\frac{d\phi}{dt}$, the angular velocity, be in the direction of the motion of the hands of a watch. Let v be the initial velocity of translation of the centre of the sphere, and ω the initial angular velocity.

(1) The equations of motion are,

$$M\frac{d^2x}{dt^2} = Mg \sin i - F, \quad \tfrac{2}{5} Ma^2 \frac{d^2\phi}{dt^2} = Fa,$$

and the condition for pure rolling is

$$\frac{dx}{dt} - a\frac{d\phi}{dt} = 0.$$

If t_0 be the time at which pure rolling begins, then

$$t_0 = \frac{2(v - a\omega)}{(7\mu \cos i - 2 \sin i)g}, \quad \text{or} \quad = \frac{2(a\omega - v)}{(7\mu \cos i + 2 \sin i)g},$$

according as $v > a\omega$, or $a\omega > v$, where μ is the coefficient of *dynamical* friction. If $v - a\omega > 0$, $7\mu \cos i > 2 \sin i$, in order that pure rolling should be attainable. If $v - a\omega = 0$, pure rolling will continue, provided $7\mu' \cos i > 2 \sin i$ (where μ' is the coefficient of *statical* friction). If $v - a\omega < 0$, pure rolling will be reached necessarily, and will then continue, provided $7\mu' \cos i > 2 \sin i$.

If $\left(\frac{dx}{dt}\right)_0$ and $\left(\frac{d\phi}{dt}\right)_0$ be the values of $\frac{dx}{dt}$ and $\frac{d\phi}{dt}$ when pure rolling is attained,

$$a\left(\frac{d\phi}{dt}\right)_0 = \left(\frac{dx}{dt}\right)_0 = \frac{5\mu v \cos i - 2(\sin i - \mu \cos i)a\omega}{7\mu \cos i - 2 \sin i},$$

or

$$a\left(\frac{d\phi}{dt}\right)_0 = \left(\frac{dx}{dt}\right)_0 = \frac{5\mu v \cos i + 2(\sin i + \mu \cos i)a\omega}{7\mu \cos i + 2 \sin i},$$

according as $v - a\omega$ is positive or negative. After pure rolling begins, if it continues, $F = \frac{2}{7} Mg \sin i$,

$$\frac{dx}{dt} = a\frac{d\phi}{dt} = \tfrac{5}{7}(t - t_0)\, g \sin i + a\left(\frac{d\phi}{dt}\right)_0.$$

(2) The equations of motion are

$$M\frac{d^2x}{dt^2} = Mg \sin i - F, \qquad \tfrac{2}{5} Ma^2 \frac{d^2\phi}{dt^2} = Fa - fMg \cos i.$$

Hence, according as $v > a\omega$, or $a\omega > v$, ω being positive,

$$t_0 = \frac{2(v - a\omega)}{\left\{\left(7\mu - 5\frac{f}{a}\right)\cos i - 2 \sin i\right\} g}, \text{ or } = \frac{2(a\omega - v)}{\left\{\left(7\mu + 5\frac{f}{a}\right)\cos i + 2 \sin i\right\} g}.$$

In the former case we must have

$$\left(7\mu - 5\frac{f}{a}\right)\cos i > 2 \sin i,$$

in order that pure rolling may be possible.

Again
$$\left(\frac{dx}{dt}\right)_0 = a\left(\frac{d\phi}{dt}\right)_0 = \frac{5\left(\mu - \frac{f}{a}\right) v \cos i - 2(\sin i - \mu \cos i)\, a\omega}{\left(7\mu - 5\frac{f}{a}\right)\cos i - 2 \sin i},$$

or
$$= \frac{5\left(\mu + \frac{f}{a}\right) v \cos i + 2(\sin i + \mu \cos i)\, a\omega}{\left(7\mu + 5\frac{f}{a}\right)\cos i + 2 \sin i},$$

and, at any time after pure rolling is established,

$$a\frac{d\phi}{dt} = a\left(\frac{d\phi}{dt}\right)_0 + \tfrac{5}{7} g\left(\sin i - \frac{f}{a}\cos i\right)(t - t_0).$$

If ω be negative, that is, in a direction opposite to the motion of the hands of a watch, the equations of motion are

$$M\frac{d^2x}{dt^2} = Mg \sin i - \mu Mg \cos i,$$

$$\tfrac{2}{5} Ma^2 \frac{d^2\phi}{dt^2} = a\mu Mg \cos i + fMg \cos i,$$

until $\frac{d\phi}{dt} = 0$. This takes place at a time

$$t_1 = \tfrac{2}{5}\,\frac{a\omega'}{g\cos i\left(\mu + \frac{f}{a}\right)}$$

where $\omega' = -\omega$.

Then
$$\left(\frac{dx}{dt}\right)_1 = \frac{2a\omega'(\sin i - \mu\cos i) + 5v\cos i\left(\mu + \frac{f}{a}\right)}{5\cos i\left(\mu + \frac{f}{a}\right)}.$$

After this $\frac{d\phi}{dt}$ is positive, and

$$t_0 = t_1 + \frac{2\left(\frac{dx}{dt}\right)_1}{\left\{\left(7\mu - 5\frac{f}{a}\right)\cos i - 2\sin i\right\}g}$$

$$= 2\,\frac{\left(\mu - \frac{f}{a}\right)a\omega' + \left(\mu + \frac{f}{a}\right)v}{\left(\mu + \frac{f}{a}\right)\left\{\left(7\mu - 5\frac{f}{a}\right)\cos i - 2\sin i\right\}g},$$

and

$$\left(\frac{dx}{dt}\right)_0 = a\left(\frac{d\phi}{dt}\right)_0 = \frac{5\left(\mu - \frac{f}{a}\right)v\cos i + 2\frac{\mu - \frac{f}{a}}{\mu + \frac{f}{a}}(\sin i - \mu\cos i)\,a\omega'}{\left(7\mu - 5\frac{f}{a}\right)\cos i - 2\sin i}.$$

17. A uniform bar, resting on a smooth horizontal table, revolves round a vertical axis through its middle point. The bar suddenly snaps at its middle point. Determine the subsequent motion of the parts.

18. In the same case, find the point of its length, at which either half of the bar would strike perpendicularly against a fixed obstacle with the greatest force of percussion.

CHAPTER X.

MOTION OF A RIGID BODY IN GENERAL.

SECTION I.—*Kinematics.*

229. **Motion of a Body having one Point fixed.**—If a rigid body have a fixed point, a spherical surface S fixed in the body, with this point as centre, must move about on the surface of an equal concentric sphere fixed in space. The position in space of S, or of any definite great circle on it, determines that of the body. Hence the motion of a body having a fixed point is reducible to the motion of a spherical figure on a sphere fixed in space. The position of such a figure is determined by the positions of any two definite points A and B in it. If the points A and B move into new positions A' and B', arcs of great circles bisecting AA' and BB' at right angles will meet in a point O, and the angle $AOA' = BOB'$; but the great circle OA can be moved into the position OA' by turning it through the angle AOA' round the axis CO (C being the centre of the sphere); and since $AOA' = BOB'$, the same rotation brings OB into the position OB'. Hence a rotation round OC brings the spherical figure, of which A and B are definite points, from the first position into the second. The point O is called the pole of rotation (*Differential Calculus*, Art. 300).

Consequently, *a rigid body having a point fixed can be moved from any one position into any other by a rotation round an axis through the point.*

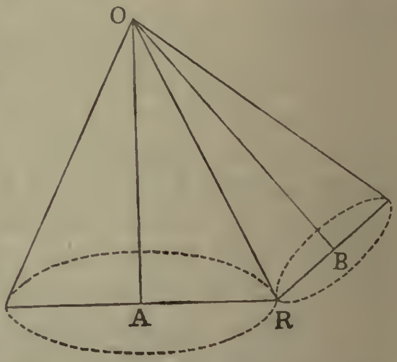
230. **Composition of Rotations round Axes meeting in a Point.**—If a body receive rotational displacements round two axes fixed in space, passing through the same point, the resultant displacement may be effected by a rotation round a single axis.

If the displacements be infinitely small, it appears, as in Article 196, that the order in which they are effected is indifferent, and also that it is indifferent whether the axes be fixed in space or be axes fixed in the body, whose positions at the commencement of the infinitely small motion coincide with those of the axes fixed in space. If the two displacements be regarded as simultaneous, the resultant rotation is the actual motion of the body. Hence we see that—

A velocity of rotation round a single axis is equivalent to velocities of rotation round two axes meeting the axis of the resultant rotation in the same point.

Being given the velocities of rotation of a rigid body round two axes meeting in a point, to determine the velocity of the resultant rotation and the position of its axis.

Let OA and OB be the axes of the component rotations, and R a point on the axis of the resultant rotation. As R is at rest during the motion, its displacement from the rotation round OA must be equal and opposite to that from the rotation round OB. Hence the circles passing through R, and having their planes at right angles to OA and OB, and their centres on those lines, touch at R. Hence OA, OB, and OR lie in the same plane. This appears readily from the fact, that if two small circles of a sphere touch, the arc of a great circle joining their poles passes through the point of contact. Again, AR multiplied by the angular velocity round OA is equal and opposite to BR multiplied by the angular velocity round OB. If these angular velocities be denoted by α and β, we have



$$\frac{\alpha}{\sin BOR} = \frac{\beta}{\sin AOR}.$$

To find ω, the angular velocity of the resultant rotation, consider the motion of A. It is unaffected by the rotation round OA, and may be regarded indifferently, as rotating round OB with angular velocity β, or as rotating round OR with angular velocity ω.

If perpendiculars AP and AQ be let fall on OB and OR, we have then $AP \,.\, \beta = AQ \,.\, \omega$. Hence

$$\frac{\omega}{\sin AOB} = \frac{\beta}{\sin AOR} = \frac{a}{\sin BOR}.$$

Hence, finally—*The axis of the resultant rotation lies in the same plane as the axes of the component rotations, and makes with each an angle whose sine is proportional to the velocity of rotation round the other; and the velocity of the resultant rotation is proportional to the sine of the angle between the axes of the component rotations.*

Accordingly, velocities of rotation are compounded in precisely the same manner as velocities of translation, or as forces meeting in a point.

By reversing the reasoning above, it can be shown that a point R, taken as above, remains at rest under the influence of two velocities of rotation round OA and OB; whence we have an independent proof, that infinitely small rotations round two intersecting axes are equivalent to a single one round an axis lying in the plane of the two former, and passing through their point of intersection.

We have already seen, Article 196, that velocities of rotation round parallel axes are compounded in the same way as parallel forces. Hence, in general—*Velocities of rotation are compounded like forces whose directions coincide with the axes of rotation, and whose magnitudes are proportional to the magnitudes of the velocities of rotation.*

The attention of the reader has been directed in Article 196 to the algebraical signs of velocities of rotation. In addition to what was there stated, it may be observed, that the axis of a rotation may be made to represent the rotation both in magnitude and direction. In this case the axis is drawn so that the rotation round it is always positive. For example, instead of speaking of a negative rotation round the axis of X, we may designate it simply as a rotation round the axis of X negative. When the axis of a rotation determines the direction of the rotation, the latter is always understood to be in the positive direction round this axis, that is, according to the convention, counter-clockwise.

When rotations are compounded by means of their axes, like forces, the direction of the axis determines in this way the direction of the rotation.

For example, rotations ω_1, ω_2, ω_3 round three rectangular axes produce a resultant rotation ω which is always positive; but the direction of its axis is determined by $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$, the cosines of the angles made with the coordinate axes; and these again depend on the signs of ω_1, ω_2, and ω_3, as well as on their magnitudes.

231. **Geometrical representation of the Motion of a Body having a fixed Point.**—When a body has a fixed point, its motion may be represented in a manner analogous to that mentioned in Article 199. In the present case the curves which correspond to the space centrode and the body centrode are spherical curves lying on the surface of the same sphere.

The motion of the body is represented by the rolling of a cone fixed in the body on a cone fixed in space (see *Differential Calculus*, Article 301).

232. **Motion of a Body which is entirely Free.**—*A rigid body can be moved from any one position into any other by a motion of translation, combined with a motion of rotation round an axis through any arbitrary point A of the body.*

Let A_1, A_2 be the two positions in space occupied by A in the different positions of the body. Give to every point of the body a motion equal and parallel to A_1A_2; this brings A into the required position, and a rotation round an axis through A will then (Article 229) complete the body's change of place.

If two positions of a body in motion are infinitely near each other, any infinitely small displacements, by which it can be moved from the first of these positions to the second, may be regarded as the actual motion of the body.

The actual motion of a rigid body during an infinitely short time is, therefore, a motion of translation together with a motion of rotation round an axis through any arbitrary point of the body.

The initial and final positions of a body being given, the magnitude of the rotation, which is required to make it pass from one to *the other, and the direction of its axis are determined; but the motion of translation varies according to the point through which the axis of rotation is supposed to pass.*

First, let the axis of rotation be supposed to pass through a point A, whose initial and final positions are A_1, A_2. The motion of translation A_1A_2 is composed of two parts, one A_1A' in the direction of the axis of rotation through A, and the other $A'A_2$ at right angles to it. By means of the first a definite plane section of the body, passing through A and at right angles to the axis of rotation, is moved into the plane in space in which it lies in its final position, and the subsequent motion of the body is therefore parallel to this plane. If, now, the axis of rotation be regarded as passing through another point B of the body, whose initial and final positions are B_1, B_2, we can suppose the translation B_1B_2 made up of two parts—one, B_1B', equal and parallel to A_1A'; the other, $B'B_2$, which depends on the position of the point. B_1B' brings the body into the same position as A_1A'. Hence, a translation $B'B_2$ and a rotation round an axis through B are equivalent to an equal rotation round a parallel axis through A and a translation $A'A_2$ (Art. 195). The translation B_1B_2 is the resultant of B_1B' and $B'B_2$; A_1A_2 is the resultant of A_1A' and $A'A_2$; B_1B' is equal and parallel to A_1A'; but $B'B_2$ is not in general either equal or parallel to $A'A_2$.

233. **Analytical Treatment of the Motion of a Body having a Fixed Point.**—Suppose three rectangular axes fixed in the body, passing through a point O; and three others fixed in space, which at the beginning of the motion coincide with the former. Let the coordinates of any point of the body referred to the former be ξ, η, ζ, and referred to the latter, x, y, z. Let the cosines of the angles which ξ makes with x, y, z be a_1, a_2, a_3; of those which η makes, b_1, b_2, b_3; and of those which ζ makes, c_1, c_2, c_3; and the angles themselves α_1, α_2, α_3; β_1, β_2, β_3; γ_1, γ_2, γ_3.

If the point O remain fixed we have at any instant

$$x = a_1\xi + b_1\eta + c_1\zeta, \quad y = a_2\xi + b_2\eta + c_2\zeta, \quad z = a_3\xi + b_3\eta + c_3\zeta.$$

If at this instant any other point of the body besides O occupy the same position in space as at the beginning of the motion, for this point, $x = \xi$, $y = \eta$, $z = \zeta$, and therefore we should have

$$\begin{vmatrix} a_1 - 1 & b_1 & c_1 \\ a_2 & b_2 - 1 & c_2 \\ a_3 & b_3 & c_3 - 1 \end{vmatrix} = 0.$$

But it is easy to see that this condition is fulfilled, for putting

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta,$$

and denoting the former determinant by Δ', we have

$$\Delta' = \Delta - \left(\frac{d\Delta}{da_1} + \frac{d\Delta}{db_2} + \frac{d\Delta}{dc_3}\right) + \frac{d^2\Delta}{da_1\,db_2} + \frac{d^2\Delta}{db_2\,dc_3} + \frac{d^2\Delta}{dc_3\,da_1} - \frac{d^3\Delta}{da_1\,db_2\,dc_3}.$$

Now $\Delta = 1$, $\dfrac{d\Delta}{da_1} = a_1$, &c., $\dfrac{d^2\Delta}{da_1\,db_2} = c_3$, &c., $\dfrac{d^3\Delta}{da_1\,db_2\,dc_3} = 1$;

therefore $\Delta' = 0$. This result can also be seen immediately by the multiplication of the determinants, which gives

$$\Delta\Delta' = -\Delta'.$$

Hence we conclude that if a rigid body have a fixed point, any two positions have a line in common.

Again, if O be fixed,

$$dx = \xi\,da_1 + \eta\,db_1 + \zeta\,dc_1, \quad dy = \xi\,da_2 + \eta\,db_2 + \zeta\,dc_2,$$

$$dz = \xi\,da_3 + \eta\,db_3 + \zeta\,dc_3;$$

but, since at the beginning of the motion ξ, η, ζ coincide with x, y, z, we have at that instant

$$a_1 = \cos \alpha_1, \quad da_1 = -\sin \alpha_1\,d\alpha_1 = 0, \quad \text{since} \quad \alpha_1 = 0.$$

In like manner $db_2 = 0$, $dc_3 = 0$;

also $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0.$

Differentiating, and remembering that initially

$$a_1 = 1,\quad a_2 = 0,\quad a_3 = 0,\quad b_1 = 0,\quad b_2 = 1,\quad b_3 = 0,$$

we have $$db_1 + da_2 = 0.$$

In like manner $dc_1 + da_3 = 0,\quad db_3 + dc_2 = 0.$

Let now $da_2 = d\psi,\quad db_3 = d\theta,\quad dc_1 = d\phi;$

then
$$\left.\begin{aligned} dx &= -\eta\, d\psi + \zeta\, d\phi = -y\, d\psi + z\, d\phi \\ dy &= -\zeta\, d\theta + \xi\, d\psi = -z\, d\theta + x\, d\psi \\ dz &= -\xi\, d\phi + \eta\, d\theta = -x\, d\phi + y\, d\theta \end{aligned}\right\}. \qquad (1)$$

But a rotation $d\theta$ round x would give (Art. 197),

$$dy = -\zeta d\theta,\quad dz = \eta d\theta;$$

$d\phi$ round y would give

$$dz = -\xi\, d\phi,\quad dx = \zeta d\phi;$$

and $d\psi$ round z would give

$$dx = -\eta d\psi,\quad dy = \xi d\psi.$$

Hence the most general infinitely small displacement the body can take, O remaining fixed, is equivalent to rotations round any three rectangular axes through O.

Moreover, from the values of dx, dy, dz, given above, it appears that for a point whose coordinates fulfil the conditions $\frac{\xi}{d\theta} = \frac{\eta}{d\phi} = \frac{\zeta}{d\psi}$ the displacements are zero.

Hence the three rotations $d\theta, d\phi, d\psi$, round the axes x, y, z, are equivalent to a single rotation round an axis whose position is defined by these equations. If we put

$$d\theta = d\omega \cos\lambda,\quad d\phi = d\omega \cos\mu,\quad d\psi = d\omega \cos\nu,$$

where $$d\omega = \sqrt{d\theta^2 + d\phi^2 + d\psi^2},$$

the equations of the fixed axis are

$$\frac{\xi}{\cos\lambda} = \frac{\eta}{\cos\mu} = \frac{\zeta}{\cos\nu}.$$

Also, for any point of the body,

$$dx^2 + dy^2 + dz^2 = \{(\eta \cos \nu - \zeta \cos \mu)^2 + (\zeta \cos \lambda - \xi \cos \nu)^2$$
$$+ (\xi \cos \mu - \eta \cos \lambda)^2\} d\omega^2 = p^2 d\omega^2,$$

if p be the perpendicular from the point on the fixed axis. Hence $d\omega$ is the magnitude of the resultant rotation.

Whence infinitely small rotations, and therefore velocities of rotation, are compounded like forces meeting at a point.

234. **Motion of a Body entirely Free.**—If the point of intersection of the axes fixed in the body be not fixed in space, and if its coordinates, referred to axes fixed in space, be x', y', z'; then, for any point x, y, z of the body,

$$x = x' + a_1\xi + b_1\eta + c_1\zeta, \quad y = y' + a_2\xi + b_2\eta + c_2\zeta,$$
$$z = z' + a_3\xi + b_3\eta + c_3\zeta;$$

whence

$$dx = dx' + \xi da_1 + \eta db_1 + \zeta dc_1, \ dy = dy' + \xi da_2 + \eta db_2 + \zeta dc_2,$$
$$dz = dz' + \xi da_3 + \eta db_3 + \zeta dc_3.$$

If we suppose the axes of ξ, η, ζ parallel to those of x, y, z at the beginning of the motion, we get, as in the last Article,

$$\left.\begin{aligned} dx &= dx' - \eta d\psi + \zeta d\phi = dx' - (y - y') d\psi + (z - z') d\phi \\ dy &= dy' - \zeta d\theta + \xi d\psi = dy' - (z - z') d\theta + (x - x') d\psi \\ dz &= dz' - \xi d\phi + \eta d\theta = dz' - (x - x') d\phi + (y - y') d\theta \end{aligned}\right\}, \quad (2)$$

and we see that

The most general infinitely small displacement which a rigid body can receive consists of a movement of translation, and a movement of rotation round an axis through any arbitrary point of the body.

Whatever be the point through which the axis of rotation is supposed to pass, the direction and magnitude of the rotation remain unaltered.

Suppose two points whose coordinates are $x', y', z', x'', y'', z''$, successively regarded as the points through which the axis of rotation passes; then,

$$dx = dx'' - (y - y'')\,d\psi'' + (z - z'')\,d\phi'';$$

also
$$dx' = dx'' - (y' - y'')\,d\psi'' + (z' - z'')\,d\phi''.$$

Subtracting, we get

$$dx = dx' - (y - y')\,d\psi'' + (z - z')\,d\phi'';$$

but again,

$$dx = dx' - (y - y')\,d\psi' + (z - z')\,d\phi'.$$

Comparing these, we see that

$$d\psi' = d\psi'', \quad d\phi' = d\phi'';$$

therefore, &c.

235. **Velocity of any Point of a Body.**—Infinitely small displacements divided by the element of time during which they are effected become velocities. If the axes of x, y, z be three rectangular axes fixed in space, and if the velocities of rotation round parallel axes meeting at the point x', y', z', be ω_x, ω_y, ω_z, we have, from equations (2),

$$\left.\begin{aligned}\frac{dx}{dt} &= \frac{dx'}{dt} - (y - y')\,\omega_z + (z - z')\,\omega_y\\ \frac{dy}{dt} &= \frac{dy'}{dt} - (z - z')\,\omega_x + (x - x')\,\omega_z\\ \frac{dz}{dt} &= \frac{dz'}{dt} - (x - x')\,\omega_y + (y - y')\,\omega_x\end{aligned}\right\}. \qquad (3)$$

If the point x', y', z' be fixed in space, as well as in the body, and be taken for the origin, we have

$$\left.\begin{aligned}\frac{dx}{dt} &= -\,y\omega_z + z\omega_y\\ \frac{dy}{dt} &= -\,z\omega_x + x\omega_z\\ \frac{dz}{dt} &= -\,x\omega_y + y\omega_x\end{aligned}\right\}. \qquad (4)$$

If we suppose the axes fixed in space to coincide at the instant under consideration with axes fixed in the body, and if the angular velocities round the latter be ω_1, ω_2, ω_3, we have $\omega_x = \omega_1$, $\omega_y = \omega_2$, $\omega_z = \omega_3$. If we call the coordinates of any point, referred to the axes fixed in the body, ξ, η, ζ, and the components of its velocity parallel to these axes u, v, w, we have

$$\left.\begin{aligned} u &= -\eta\omega_3 + \zeta\omega_2 \\ v &= -\zeta\omega_1 + \xi\omega_3 \\ w &= -\xi\omega_2 + \eta\omega_1 \end{aligned}\right\}. \qquad (5)$$

Equations (3), (4), and (5) hold good for every instant, whereas the equations $x = \xi$, &c., $\omega_x = \omega_1$, &c., $\dfrac{dx}{dt} = u$, &c., hold good only for one particular instant.

236. **Accelerations of Rotations.**—If ω_1, ω_2, ω_3, be the angular velocities round three rectangular axes, OA, OB, OC fixed in the body, and ω_x, ω_y, ω_z the velocities round axes OX, OY, OZ fixed in space; and if at any instant we suppose OX, OY, OZ to coincide with the positions occupied at the instant by OA, OB, OC, not only is ω_1 equal to ω_x, ω_2 to ω_y, and ω_3 to ω_z, but also

$$\frac{d\omega_1}{dt} = \frac{d\omega_x}{dt}, \quad \frac{d\omega_2}{dt} = \frac{d\omega_y}{dt}, \quad \frac{d\omega_3}{dt} = \frac{d\omega_z}{dt}.$$

Let ω be the velocity of rotation round a line fixed in the body, which passes through O, and makes angles α, β, γ with the axes OX, OY, OZ; then

$$\omega = \omega_x \cos\alpha + \omega_y \cos\beta + \omega_z \cos\gamma;$$

therefore
$$\frac{d\omega}{dt} = \frac{d\omega_x}{dt}\cos\alpha + \frac{d\omega_y}{dt}\cos\beta + \frac{d\omega_z}{dt}\cos\gamma$$

$$- \omega_x \sin\alpha \frac{d\alpha}{dt} - \omega_y \sin\beta \frac{d\beta}{dt} - \omega_z \sin\gamma \frac{d\gamma}{dt}.$$

Now, if the line coincide with OX,

$$\alpha = 0, \quad \beta = \tfrac{1}{2}\pi, \quad \gamma = \tfrac{1}{2}\pi;$$

then
$$\frac{d\omega_1}{dt} = \frac{d\omega}{dt} = \frac{d\omega_x}{dt} - \omega_y \frac{d\beta}{dt} - \omega_z \frac{d\gamma}{dt};$$

but $\frac{d\beta}{dt}$ is the angular velocity with which a line of the body, which at the instant coincides with OX, is turning from OY towards OX, whence

$$\frac{d\beta}{dt} = -\omega_z.$$

In like manner
$$\frac{d\gamma}{dt} = \omega_y;$$

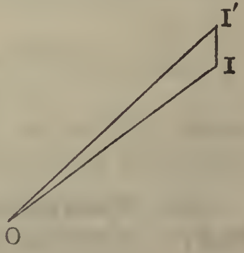
therefore, finally,
$$\frac{d\omega_1}{dt} = \frac{d\omega_x}{dt}.$$

The same may be proved geometrically as follows:—

The body at any instant is rotating round a certain axis with an angular velocity ω. Draw a line through the fixed origin in the direction of the instantaneous axis, and measure off on it a portion OI, proportional to ω, then the projections of this line on the axes fixed in space represent $\omega_x, \omega_y, \omega_z$; and its projections on the axes fixed in the body represent $\omega_1, \omega_2, \omega_3$. At the next instant the body is rotating round another line with a velocity ω', represented by OI', and the projections of OI' represent $\omega'_x, \omega'_y, \omega'_z$; $\omega'_1, \omega'_2, \omega'_3$. But the projection of OI' is equal to the sum of the projections of OI and II'. Also OI (the instantaneous axis of rotation) occupies the same position both in space and in the body as it did before. Hence,

$$d\omega_x = \omega'_x - \omega_x = \text{projection of } II' \text{ on axis of } x \text{ fixed in space},$$

$$d\omega_1 = \omega'_1 - \omega_1 = \text{projection of } II' \text{ on axis of } \xi \text{ fixed in the body}.$$



At the first instant these axes coincide; and at the next the two projections of II' differ only by a quantity infinitely small compared with II', which is itself infinitely small of the first order. Hence $d\omega_x$ and $d\omega_1$ differ by an infinitely small quantity of the second order;

or
$$\frac{d\omega_1}{dt} = \frac{d\omega_x}{dt}, \quad \frac{d\omega_2}{dt} = \frac{d\omega_y}{dt}, \quad \frac{d\omega_3}{dt} = \frac{d\omega_z}{dt}. \tag{6}$$

237. **Accelerations of a Point, parallel to three Axes fixed in the Body.**—If u, v, w be the velocities of the point parallel to axes fixed in the body, its velocity V, along a line making angles α, β, γ with these axes, is

$$u \cos \alpha + v \cos \beta + w \cos \gamma.$$

If we suppose this latter line fixed in space, the acceleration of the point parallel to it is $\frac{dV}{dt}$, and we have

$$\frac{dV}{dt} = \frac{du}{dt} \cos \alpha + \frac{dv}{dt} \cos \beta + \frac{dw}{dt} \cos \gamma$$
$$- \left(u \sin \alpha \frac{d\alpha}{dt} + v \sin \beta \frac{d\beta}{dt} + w \sin \gamma \frac{d\gamma}{dt} \right).$$

Let us now suppose

$$\alpha = 0, \quad \beta = \tfrac{1}{2}\pi, \quad \gamma = \tfrac{1}{2}\pi;$$

then
$$\frac{dV}{dt} = \frac{du}{dt} - v \frac{d\beta}{dt} - w \frac{d\gamma}{dt};$$

but $\frac{dV}{dt}$ is now the acceleration of the point parallel to one of the axes fixed in the body, whilst $\frac{d\beta}{dt}$ and $\frac{d\gamma}{dt}$ have values equal in magnitude and opposite in sign to those of Art. 236; whence we have, for the accelerations of a point parallel to three rectangular axes fixed in the body, the expressions

$$\frac{du}{dt} - v\omega_3 + w\omega_2, \quad \frac{dv}{dt} - w\omega_1 + u\omega_3, \quad \frac{dw}{dt} - u\omega_2 + v\omega_1,$$

where u, v, w are the velocities of the point parallel to the axes fixed in the body.

238. **Complete Determination of the Motion of a Body.**—Every motion which a rigid body can take is redu-

cible to a motion of translation and a motion of rotation. In order then to determine the motion of the body, a point in it is selected (usually the centre of inertia), and the motion of the body is reduced to the motion of this point, together with the rotatory motion of the body round it.

Geometrically the motion may be represented by the rolling of a cone, fixed in the body, on a movable cone unattached to the body, except at one point (the common vertex of the cones), the latter cone undergoing a motion of translation. If the two cones and the rate at which the one rolls on the other are known, as well as the position in the body of their common vertex, its velocity at each instant, and the path which it describes, then the motion of the body is completely determined.

It is usually most convenient to consider the motion of translation and the motion of rotation separately. The investigation of the former motion is, as we have seen (Art. 204), reducible to the problem of the motion of a particle. The latter motion is completely determined if we can assign at each instant the position of the body and its velocities of rotation in reference to axes, through the centre of inertia, whose directions are fixed in space.

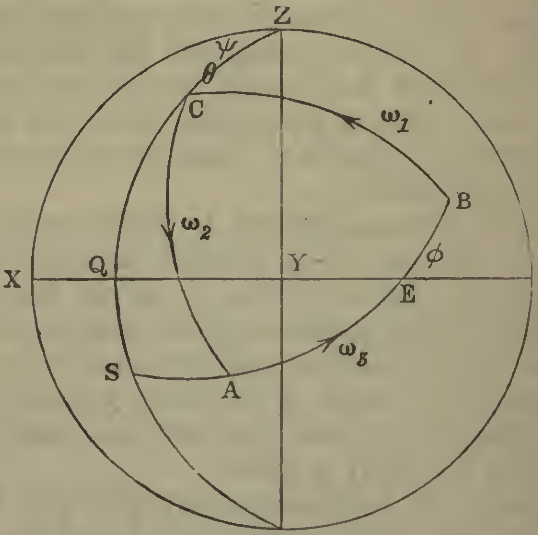
The equations of Kinetics usually give the velocities of rotation round axes fixed in the body; and in order fully to determine the motion, it is necessary to ascertain the effect of these velocities when the position of the body is referred to axes whose directions are fixed in space. As the points of intersection of these two sets of axes coincide, the velocities of rotation have no effect on the motion of this point; and therefore, so far as the present inquiry is concerned, we may regard it as fixed, not only in the body, but also in space.

Call the space-axes OX, OY, OZ; the body-axes OA, OB, OC, each set being rectangular.

Round the point O as centre describe a sphere, and let the axes meet it at the points X, Y, Z, A, B, C.

Three independent angles are required to determine the position of the body in space.

Those selected by the great mathematicians who solved the problem of Precession and Nutation are the angular coordinates of the point C, or of the line OC, and the angle which OB makes with the line of intersection of the planes AOB and XOY. It is obvious that the position of OC fixes the plane AOB, but does not determine the position of the lines OA and OB in this plane. Hence, when C is fixed, so is OE, the line of intersection of the planes XOY and AOB: if, then, the angle ϕ between OE and OB is given, the position of the body is completely determined. The angular coordinates of OC are θ, the angle which it makes with OZ, and ψ the angle which the plane COZ makes with the plane XOZ.

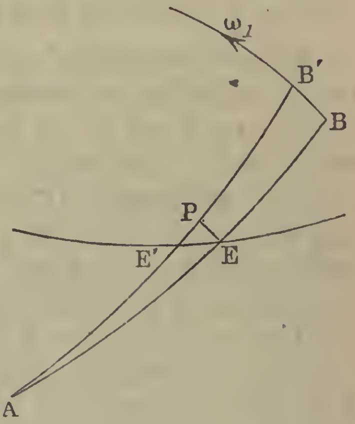


Suppose now that the body has three velocities of rotation: ω_1 round OA; ω_2 round OB; and ω_3 round OC, in the direction of the arrow heads. We have to express $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, and $\frac{d\psi}{dt}$ in terms of these velocities, remembering that the changes of θ, ϕ, and ψ are caused solely by ω_1, ω_2, ω_3. The motion of the point C on the sphere is unaffected by ω_3. If the radius of the sphere be unity, the point C has two velocities, ω_1 and ω_2, along the tangents to the great circles BC and CA. Resolving these velocities along the great circle ZC, and at right angles to it, we have

$$\frac{d\theta}{dt} = \omega_2 \cos\phi + \omega_1 \sin\phi, \tag{7}$$

$$\sin\theta \frac{d\psi}{dt} = \omega_2 \sin\phi - \omega_1 \cos\phi. \tag{8}$$

These equations are obvious, since E is the pole of ZC, and therefore the angle ACS is ϕ; and since the arc of a small circle, on a sphere whose radius is unity, is equal to the angle subtended at its pole, multiplied by the sine of the spherical radius. As regards the algebraical signs, it is well to observe that ψ is counted from ZX towards ZY; and that ϕ is positive and acute, when E lies in the quadrant AB, because ϕ is counted from OE in the direction from A to B.



If ω_3 were zero, the change in ϕ would result entirely from the motion of the plane AOB, the body having no motion parallel to that plane: ω_1 and ω_2 would then change the position of the plane AOB, or (which is the same thing) of the line OC, and consequently the position of OE. Let AB, AB' be two consecutive positions of the plane AB, resulting from the rotation ω_1 alone. Draw EP perpendicular to AB'; then $E'P = \delta\phi$. But

$$E'P = EP \cot EE'P = \omega_1 \cos\phi \cot\theta\, dt.$$

In like manner, from ω_2 alone,

$$\delta\phi = -\omega_2 \sin\phi \cot\theta\, dt.$$

If ω_1 and ω_2 were each zero, the plane AOB would be fixed, and the whole change of ϕ would result from the motion of the body parallel to this plane—in other words, from ω_3. In this case OE would be a line fixed in space, and $\delta\phi = \omega_3 dt$. Hence, adding the three displacements,

$$d\phi = \omega_3\, dt - \cot\theta (\omega_2 \sin\phi - \omega_1 \cos\phi)\, dt;$$

therefore, by (8), $$\frac{d\phi}{dt} = \omega_3 - \cos\theta\,\frac{d\psi}{dt}. \tag{9}$$

This last equation may be obtained more readily by considering the motion of the point A in the plane AB. The angular velocity of A in the plane AB is ω_3; but it is also the angular velocity of S resolved in the direction AB, together with the rate of increase of SA. Since $SE = 90°$, $SA = EB = \phi$; and since SZ and SA are at right angles, and S lies on ZC at a distance $90° + \theta$ from Z, the velocity of S along AB is $\cos\theta\,\frac{d\psi}{dt}$; therefore $\omega_3 = \cos\theta\,\frac{d\psi}{dt} + \frac{d\phi}{dt}$.

More symmetrical modes of treating the connexion between velocities of rotation round axes fixed in space, and round axes fixed in the body, will be given in the examples at the end of the section.

239. **Screws.**—It was shown in Article 232 that a body can be moved from any one position into any other by a translation combined with a rotation round an axis through any arbitrary point of the body.

The translation may be resolved into two—one parallel to the axis of rotation, and the other at right angles thereto. The latter translation, along with the rotation, may be replaced by a pure rotation round a parallel axis, and so the whole motion will consist of a translation parallel to a certain fixed line and of a rotation round it. Such a motion is similar to that of a nut on a screw, and is called a *Twist*. Hence a body can be moved from any one position into any other by means of a twist.

In order to determine a twist it is necessary to specify—(1) the magnitude of the rotation, called the *amplitude* of the twist; (2) the position and direction of the line round which the rotation is effected, or *the axis of the screw*; and (3) the ratio of the translation to the rotation. This last is a linear magnitude, and is called *the pitch of the screw*. In order, then, to determine a twist, we must specify its amplitude, and the position and pitch of the screw round which it is effected.

The twist by which a body can be moved from any one position into any other is in general unique.

This readily appears from considering that if two positions of a body are given, the magnitude of the corresponding rotation and the direction of its axis are invariable; and that if two positions of a plane figure in its own plane are given, the position of the corresponding centre of rotation is thereby determined.

The same thing is proved directly by Dr. Ball (to whom the *Theory of Screws* is principally due), as follows :—

Any point of the body, which lies on the axis of the twist, must continue thereon after the motion. If, therefore, the motion could be effected by two different twists, there would be two different lines along which points of the body would continue throughout the motion. In order that this should be possible, the lines must be parallel, and the motion one of pure translation.

If two successive positions of a body in motion are infinitely near each other, the twist by which it can be brought from the one position to the other is the actual motion of the body. We see then that the most general motion of a rigid body consists of a succession of twists. The screw round which it is twisting at any instant is called the *instantaneous screw*. As the position of a straight line in space is determined by four independent quantities, four magnitudes must be assigned to fix the position of a screw. In order to determine a twist, its amplitude, and the pitch, as well as the position, of the corresponding screw, are required. Hence the motion of a rigid body in general depends on six independent variables, and we see, as in Article 191, that a rigid body entirely unrestrained has six degrees of freedom.

240. **Composition of Twists.**—If a body receive in succession two twists whose amplitudes are infinitely small, the order in which they are effected is indifferent, and the resulting change of position may be produced by a single twist, which is the resultant of the two former.

More symmetrical results are obtained, if instead of seeking for the twist which is the resultant of two others, we inquire how three twists having infinitely small amplitudes must be related, in order that the position of a body, after being affected by them, may remain unaltered.

The question proposed may be solved directly, but the method of solution devised by Dr. Ball leads to results of a more instructive character.

Both modes of procedure will be found in the Examples.

EXAMPLES.

1. Determine the velocity with which the plane of the horizon, at a place whose latitude is given, turns round a vertical axis.

Ans. $\omega \sin \lambda$, where ω is the earth's angular velocity, and λ the latitude.

2. If the velocities of rotation of a body round three rectangular axes are given in terms of the time, show how to determine—(1) the velocity of rotation round the instantaneous axis; (2) the position of the instantaneous axis; (3) the equation of the cone which is the locus of the instantaneous axis.

3. If the velocities of rotation round three rectangular axes are proportional to the time which has elapsed from a given epoch, the position of the instantaneous axis is fixed.

4. If the accelerations of rotation round three rectangular axes are constant, the instantaneous axis lies in a fixed plane.

5. A body receives in a given order rotations of finite magnitude round two axes fixed in space, or in the body, and meeting in a point. Find the position of the axis, a single rotation round which would bring the body into the same position, and determine the magnitude of the resultant rotation.

This question is solved in a manner similar to that employed in Examples 3 and 4, Art. 200; the construction in the present case being on the surface of a sphere instead of a plane.

When the rotations round the given axes are in the same direction, the resultant rotation is double the supplement of the vertical angle of a spherical triangle, whose base and base angles are the angle between the axes and the semi-amplitudes of the rotations round them.

6. A rigid body receives a motion of translation, whose components, parallel to the axes, are a, b, c, and a rotation θ round an axis fixed in the body, which, at the beginning of the motion, coincides with the axis of z. Determine the position and pitch of the screw, a twist round which would bring the body into the same position; and find the amplitude of the twist.

The screw passes through a point whose coordinates are

$$x = \frac{a \sin \frac{1}{2}\theta - b \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta}, \qquad y = \frac{b \sin \frac{1}{2}\theta + a \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta}.$$

Pitch of screw $= \dfrac{c}{\theta}$. Amplitude of twist $= \theta$.

7. A body receives, in succession, rotations of finite magnitude round two non-intersecting axes A, B, either fixed in space or fixed in the body; if d be

the shortest distance between the lines A and B; θ and θ' the amplitudes of the rotations round them; ϵ the angle between them; ϕ the amplitude of the twist equivalent to the motion; and p the pitch of its screw; prove that

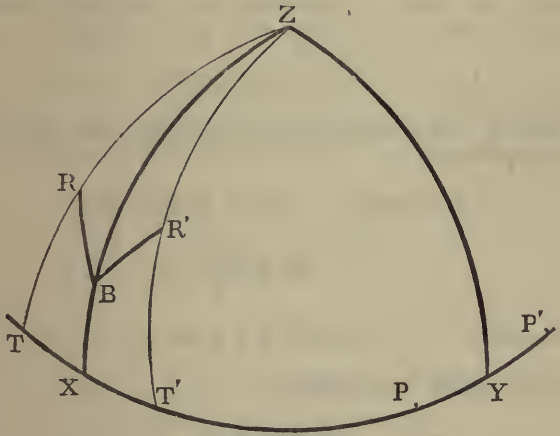
$$\tfrac{1}{2}p\phi \sin \tfrac{1}{2}\phi = d \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta' \sin \epsilon.$$

(This theorem is due to Rodrigues: *Liouville*, T. 5, p. 390.)

Take the shortest distance between A and B for axis of y; the point of intersection of this line with B for origin O; and a parallel to A for axis of z.

After the body has received its rotation round A, suppose it receives in succession two equal and opposite rotations round OZ, the first of these being equal and opposite to that round A. These rotations, being equal and opposite, do not change the position of the body. If A and B be fixed in space, so are OZ and B; if A and B be fixed in the body, but not in space, so are OZ and B.

The rotation round A, and the equal and opposite one round OZ, are (Ex. 5, Art. 200) equivalent to a translation, whose magnitude is $2d \sin \frac{1}{2}\theta$, and whose direction lies in the plane XOY, and is at right angles to a line OP or OP', which, if the axes be fixed in space, makes with OY an angle $-\frac{1}{2}\theta$; if fixed in the body, an angle $+\frac{1}{2}\theta$.



Describe a sphere round O as centre. If the axes be fixed in space, the direction of translation meets the sphere in T; and the axis of rotation, which is equivalent (Ex. 5) to the rotations round OZ and B, in R. If the axes be fixed in the body, these lines meet the sphere in T' and R', where

$$XT = XT' = \tfrac{1}{2}\theta, \quad XZT = XZT' = \tfrac{1}{2}\theta, \quad ZBR = ZBR' = \tfrac{1}{2}\theta',$$

$$BRT = BR'T' = \tfrac{1}{2}\phi, \quad ZB = \epsilon.$$

We have, then, $p\phi$ = component of translation parallel to axis of screw

$$= 2d \sin \tfrac{1}{2}\theta \cos TR \text{ (or } \cos T'R') = 2d \sin \tfrac{1}{2}\theta \sin ZR \text{ (or } \sin ZR'),$$

whence

$$\tfrac{1}{2}p\phi \sin \tfrac{1}{2}\phi = d \sin \tfrac{1}{2}\theta \sin ZR \sin \tfrac{1}{2}\phi = d \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta' \sin \epsilon.$$

8. A body receives twists, having infinitely small amplitudes, round two screws intersecting at right angles. Determine the amplitude of the resultant twist, and the position and pitch of its screw.

Take the screws for axes of x and y, let their pitches be p and q, and the amplitudes of the twists round them θ and ϕ.

The rotations θ and ϕ are equivalent to a single rotation ψ round an axis lying in the plane xy, and making an angle λ with the axis of x, where

$$\psi \cos \lambda = \theta, \quad \psi \sin \lambda = \phi.$$

The translations $p\theta$ and $q\phi$ are equivalent to a translation

$$p\theta \cos \lambda + q\phi \sin \lambda = (p \cos^2 \lambda + q \sin^2 \lambda)\psi$$

along the axis of the resultant rotation, and to a translation

$$q\phi \cos \lambda - p\theta \sin \lambda = (q - p)\psi \sin \lambda \cos \lambda$$

at right angles thereto. This translation, together with the rotation ψ round the axis through the origin, is equivalent to a rotation ψ round a parallel axis passing through a point on the axis of z, whose distance from the origin is

$$(q - p) \sin \lambda \cos \lambda.$$

Hence the position of the straight line coinciding with the resultant screw is given by the equations

$$y = x \tan \lambda, \quad z = (q - p) \sin \lambda \cos \lambda,$$

where
$$\tan \lambda = \frac{\phi}{\theta};$$

the pitch of the screw $\qquad r = p \cos^2 \lambda + q \sin^2 \lambda,$

and the amplitude of the twist being ψ,

$$\psi^2 = \theta^2 + \phi^2.$$

9. Any screw, a twist round which is the resultant of twists round two given screws intersecting at right angles, lies on a surface determined by the given screws; and its pitch depends on the angle which its direction makes with one of these screws. The amplitudes of the twists are supposed to be infinitely small.

If we eliminate λ from the equations of the last example, which define the position of the screw belonging to the resultant twist, we obtain

$$z(x^2 + y^2) - (q - p)xy = 0,$$

which is the equation of the surface. It is called the *cylindroid* by Dr. Ball, by whom it was discovered.

The pitch r of the screw is, as shown in the last example, given by the equation

$$r = p \cos^2 \lambda + q \sin^2 \lambda.$$

If we describe in the plane of xy the conic whose equation is

$$px^2 + qy^2 = \epsilon^3,$$

where ϵ is a linear constant, the square of the reciprocal of any diameter of this conic is proportional to the pitch of the parallel screw on the cylindroid. When a screw is spoken of as belonging to the cylindroid, it is understood that, not only is it one of the generating lines of the surface, but also that its pitch is defined in the manner just mentioned.

10. Prove that any two screws belong to the same cylindroid; and two screws being given, determine the cylindroid to which they belong.

Take the common perpendicular to the two screws as axis of z; we have then to determine the position of the origin and of the axis of x, and the magnitudes of the quantities p and q, so as to satisfy the equations

$$r_1 = p\cos^2\lambda_1 + q\sin^2\lambda_1, \qquad z_1 = (q-p)\sin\lambda_1\cos\lambda_1,$$

$$r_2 = p\cos^2\lambda_2 + q\sin^2\lambda_2, \qquad z_2 = (q-p)\sin\lambda_2\cos\lambda_2,$$

$$A = \lambda_1 - \lambda_2, \qquad h = z_1 - z_2,$$

where A is the angle, and h the distance between the given screws. As the number of quantities at our disposal is equal to the number of equations to be satisfied, it is always possible to determine a cylindroid containing the given screws. The equations are solved as follows:—Subtracting,

$$r_1 - r_2 = (q-p)(\sin^2\lambda_1 - \sin^2\lambda_2) = (q-p)\sin(\lambda_1+\lambda_2)\sin(\lambda_1-\lambda_2)$$

$$h = (q-p)\cos(\lambda_1+\lambda_2)\sin(\lambda_1-\lambda_2).$$

Adding,

$$r_1 + r_2 = 2p - (p-q)(\sin^2\lambda_1 + \sin^2\lambda_2) = 2q + (p-q)(\cos^2\lambda_1 + \cos^2\lambda_2)$$

$$= p + q + \tfrac{1}{2}(p-q)(\cos 2\lambda_1 + \cos 2\lambda_2)$$

$$= p + q + (p-q)\cos(\lambda_1+\lambda_2)\cos(\lambda_1-\lambda_2),$$

$$z_1 + z_2 = (q-p)\sin(\lambda_1+\lambda_2)\cos(\lambda_1-\lambda_2),$$

whence $$(q-p)\sin(\lambda_1+\lambda_2) = \frac{r_1-r_2}{\sin A}, \quad z_1 + z_2 = (r_1 - r_2)\cot A,$$

$$(q-p)\cos(\lambda_1+\lambda_2) = \frac{h}{\sin A}, \quad z_1 - z_2 = h,$$

$$p + q = r_1 + r_2 + h\cot A, \quad \lambda_1 - \lambda_2 = A,$$

and the mode of completing the solution is obvious.

11. A body receives three twists having infinitely small amplitudes. Determine the relations between the twists, in order that the position of the body should remain unaltered.

Dr. Ball's solution is as follows:—

Determine the cylindroid containing two of the screws. Take its screws intersecting at right angles for axes of x and y. Let the amplitudes of the three twists be Ω_1, Ω_2, Ω_3, and the angles which their screws make with the axis of x, λ_1, λ_2, λ_3. If the third screw belongs to the cylindroid containing the other two, and if the angle it makes with the axis of x, and the amplitude of the corresponding twist satisfy the equations

$$\frac{\Omega_1}{\sin(\lambda_2 - \lambda_3)} = \frac{\Omega_2}{\sin(\lambda_3 - \lambda_1)} = \frac{\Omega_3}{\sin(\lambda_1 - \lambda_2)},$$

the twists compensate each other.

In fact each twist can be resolved into two round the screws lying along the axes of x and y. The whole motion is thus reduced to two twists round these screws; and if the amplitudes of these twists are zero, the body remains undisturbed. But the equations above are the conditions that the rotations round the axes of x and y should be zero, and these rotations are the amplitudes of the twists.

As the twist by which a given motion can be effected is unique, there is only one twist by which two given twists can be compensated; and, therefore, if three twists compensate each other, the third screw must belong to the cylindroid containing the other two, and the above equations must hold good.

This problem may be solved directly as follows:—

If one point of a body continue at rest, and the rotations round axes through it compensate each other, the body must remain undisturbed. Let the three axes of rotation be transferred to the origin. In order that the rotations should compensate each other, the three axes, when transferred, must lie in the same plane. Hence the three screws are at right angles to the same line. Take it for axis of z, and let the coordinates of its intersections with the screws be z_1, z_2, z_3, and the pitches of the screws p_1, p_2, p_3: then, in order that the origin should have no motion of translation,

$$p_1\Omega_1\cos\lambda_1 + p_2\Omega_2\cos\lambda_2 + p_3\Omega_3\cos\lambda_3 - z_1\Omega_1\sin\lambda_1 - z_2\Omega_2\sin\lambda_2 - z_3\Omega_3\sin\lambda_3 = 0.$$

$$p_1\Omega_1\sin\lambda_1 + p_2\Omega_2\sin\lambda_2 + p_3\Omega_3\sin\lambda_3 + z_1\Omega_1\cos\lambda_1 + z_2\Omega_2\cos\lambda_2 + z_3\Omega_3\cos\lambda_3 = 0;$$

and, in order that the rotations round axes through the origin should compensate each other,

$$\Omega_1\cos\lambda_1 + \Omega_2\cos\lambda_2 + \Omega_3\cos\lambda_3 = 0,$$

$$\Omega_1\sin\lambda_1 + \Omega_2\sin\lambda_2 + \Omega_3\sin\lambda_3 = 0.$$

From the last two we get

$$\Omega_3^2 = \Omega_1^2 + \Omega_2^2 + 2\Omega_1\Omega_2\cos(\lambda_1 - \lambda_2),$$

$$\tan\lambda_3 = \frac{\Omega_1\sin\lambda_1 + \Omega_2\sin\lambda_2}{\Omega_1\cos\lambda_1 + \Omega_2\cos\lambda_2}.$$

Eliminating p_3 from the first two, we get

$$\Omega_3 z_3 = \Omega_1 \{p_1 \sin(\lambda_3 - \lambda_1) - z_1 \cos(\lambda_3 - \lambda_1)\} + \Omega_2 \{p_2 \sin(\lambda_3 - \lambda_2) - z_2 \cos(\lambda_3 - \lambda_2)\}.$$

If we substitute for $\cos\lambda_3$ and $\sin\lambda_3$ their values, we obtain

$$\Omega_3^2 z_3 = \Omega_1^2 z_1 + \Omega_2^2 z_2 + \Omega_1 \Omega_2 \{(z_1 + z_2)\cos(\lambda_1 - \lambda_2) + (p_1 - p_2)\sin(\lambda_1 - \lambda_2)\};$$

and, in like manner,

$$\Omega_3^2 p_3 = \Omega_1^2 p_1 + \Omega_2^2 p_2 + \Omega_1 \Omega_2 \{(p_1 + p_2)\cos(\lambda_1 - \lambda_2) - (z_1 - z_2)\sin(\lambda_1 - \lambda_2)\}.$$

12. A body is moving round a fixed point. Determine the accelerations of any point parallel and at right angles to the instantaneous axis of rotation.

Taking three lines fixed in space through the fixed point as axes,

$$\frac{dx}{dt} = \omega_y z - \omega_z y, \quad \frac{dy}{dt} = \omega_z x - \omega_x z, \quad \frac{dz}{dt} = \omega_x y - \omega_y x;$$

differentiating, and substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, from these equations, we obtain

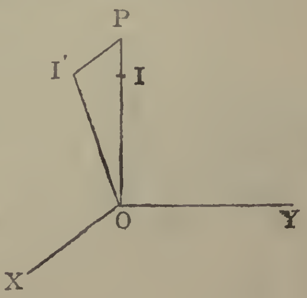
$$\frac{d^2 x}{dt^2} = \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + z\frac{d\omega_y}{dt} - y\frac{d\omega_z}{dt},$$

$$\frac{d^2 y}{dt^2} = \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + x\frac{d\omega_z}{dt} - z\frac{d\omega_x}{dt},$$

$$\frac{d^2 z}{dt^2} = \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + y\frac{d\omega_x}{dt} - x\frac{d\omega_y}{dt},$$

remembering that $$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2.$$

Let us now suppose the axis of z to coincide with the instantaneous axis, then $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$. Let the plane of xz pass through the next consecutive position of the instantaneous axis; that is be a tangent plane to the cone C fixed in space, the locus of the instantaneous axis. Measure off OI proportional to ω on OZ, let OI' be the consecutive position of the instantaneous axis, and take OI' proportional to the corresponding angular velocity $\omega + d\omega$; draw $I'P$ perpendicular to OI; then $\omega + d\omega$ round OI' is resolvable into OP round OZ, and $I'P$ round OX.



Let $I'OP = d\sigma$, then $I'P = OI'\,d\sigma$, therefore

$$\frac{d\omega_x}{dt} = \omega\frac{d\sigma}{dt} = \omega\psi,$$

if the angular velocity of the instantaneous axis be denoted by ψ. Also

$$\frac{d\omega_y}{dt} = 0, \qquad \frac{d\omega_z}{dt} = \frac{d\omega}{dt},$$

since

$$d\omega_z \propto IP = OI' - OI.$$

Introducing these, we obtain

$$\frac{d^2 x}{dt^2} = -\omega^2 x - \frac{d\omega}{dt} y,$$

$$\frac{d^2 y}{dt^2} = \frac{d\omega}{dt} x - \omega^2 y - \omega\psi z,$$

$$\frac{d^2 z}{dt^2} = \omega\psi y.$$

13. Find the position of the acceleration-centre in a body rotating round a fixed point.

The only acceleration-centre which in general exists is the fixed point itself.

14. A body is rotating round a fixed point: find the locus of a point whose acceleration along its path at any given instant is zero.

As the path at the instant touches a circle, having its centre on the instantaneous axis and its plane at right angles thereto, if r be the distance of any point from the axis, S, the tangential acceleration,

$$= -\frac{y}{r}\frac{d^2 x}{dt^2} + \frac{x}{r}\frac{d^2 y}{dt^2} = \frac{x^2 + y^2}{r}\frac{d\omega}{dt} - \omega\psi\frac{xz}{r}.$$

The required locus is therefore the cone

$$\frac{d\omega}{dt}(x^2 + y^2) - \omega\psi xz = 0.$$

15. Show that a point whose normal acceleration at right angles to the instantaneous axis vanishes lies on the cone

$$\omega (x^2 + y^2) + \psi yz = 0.$$

16. A body is rotating round a fixed point: determine at any instant the positions of the osculating plane, and of the principal normal to the path described by one of its points.

A plane passing through the point, and the instantaneous axis, is a normal plane to the path. Hence the perpendicular to the osculating plane is the intersection of two planes, the first of which passes through the instantaneous axis and the point; the second, through the next consecutive positions of the instantaneous axis and the point. The direction of the principal normal coincides with that of the resultant normal acceleration; hence, if ν be the angle it makes with the instantaneous axis,

$$\tan \nu = -\frac{\omega r^2 + \psi yz}{\psi r y}.$$

17. Find the radius of curvature of the path of a point.

If N be whole normal acceleration,

$$\rho = \frac{\omega^2 r^2}{N} = \frac{\omega r^3}{\surd\{(\omega r^2 + \psi y z)^2 + \psi^2 y^2 r^2\}}.$$

18. A body is moving in any manner. Determine the accelerations of a point parallel and at right angles to the axis of the instantaneous screw.

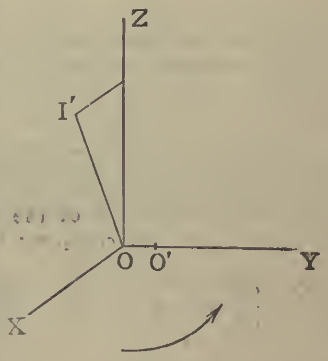
Let x_0, y_0, z_0 be the coordinates of a point fixed in the body, ξ, η, ζ the coordinates of any point referred to x_0, y_0, z_0 as origin; then

$$\frac{d^2 x}{dt^2} = \frac{d^2 x_0}{dt^2} + \omega_x(\omega_x \xi + \omega_y \eta + \omega_z \zeta) - \omega^2 \xi + \zeta \frac{d\omega_y}{dt} - \eta \frac{d\omega_z}{dt},$$

$$\frac{d^2 y}{dt^2} = \frac{d^2 y_0}{dt^2} + \omega_y(\omega_x \xi + \omega_y \eta + \omega_z \zeta) - \omega^2 \eta + \xi \frac{d\omega_z}{dt} - \zeta \frac{d\omega_x}{dt},$$

$$\frac{d^2 z}{dt^2} = \frac{d^2 z_0}{dt^2} + \omega_z(\omega_x \xi + \omega_y \eta + \omega_z \zeta) - \omega^2 \zeta + \eta \frac{d\omega_x}{dt} - \xi \frac{d\omega_y}{dt}.$$

Take as x_0, y_0, z_0 that point of the body which at the instant coincides with the point on the instantaneous screw in space which is nearest the consecutive position of the instantaneous screw. If C be the ruled surface in space, generated by the positions of the instantaneous screw-axis, x_0, y_0, z_0 will coincide with the point of intersection of the instantaneous screw-axis, and the line of striction on C. Let O be this point, and O' the point in which the shortest distance between the consecutive positions of the instantaneous screw-axis meets the consecutive screw-axis. At the time $t + dt$ the body is twisting round a screw through O'. Let T be the velocity of translation at the time t, and T' and ω' the velocities of translation and rotation at the time $t + dt$.



Now, the rotation ω' round $O'S$ (screw-axis through O') is equivalent to ω' round OI' (parallel to $O'S$), and a translation $\omega' . OO'$ at right angles to OO' and OI'.

The velocity of rotation ω' round OI' is also equivalent to ω' round OZ, and $\omega' d\sigma$ round OX. Hence, at the time $t + dt$, the point x_0, y_0, z_0 has two velocities of translation: T' along OZ, and $(\omega' . OO' + T' d\sigma)$ along OX. Again, as OO' is infinitely small of the first order, being equal to $U dt$, the velocity of translation along OZ, resulting from $OO' . \omega'$, is infinitely small of the second order. At the time t the point x_0, y_0, z_0 had the velocity T along OZ. Hence, if U be the velocity of translation, and ψ the angular velocity of the axis of the instantaneous screw, at the instant, we have

$$\frac{d\sigma}{dt} = \psi, \quad \frac{d^2 x_0}{dt^2} = \omega U + T\psi, \quad \frac{d^2 y_0}{dt^2} = 0, \quad \frac{d^2 z_0}{dt^2} = \frac{dT}{dt}.$$

Also $\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = \omega, \quad \frac{d\omega_x}{dt} = \omega\psi, \quad \frac{d\omega_y}{dt} = 0, \quad \frac{d\omega_z}{dt} = \frac{d\omega}{dt};$

whence
$$\frac{d^2x}{dt^2} = \omega U + T\psi - \omega^2\xi - \frac{d\omega}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = \frac{d\omega}{dt}\xi - \omega^2\eta - \omega\psi\zeta,$$

$$\frac{d^2z}{dt^2} = \frac{dT}{dt} + \omega\psi\eta.$$

19. A body is moving in any way: determine the position of the acceleration-centre.

Its coordinates are formed from the equations of the last example, by making

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} = 0.$$

20. A body is moving in any way: the acceleration of any point is the same as if the body were rotating round the acceleration-centre as an absolutely fixed point.

Express the accelerations, by the last example, in terms of the coordinates relative to the acceleration-centre as origin, and the results are the same as the expressions of Ex. 12 would become, if we made

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = \omega, \quad \frac{d\omega_x}{dt} = \omega\psi, \quad \frac{d\omega_y}{dt} = 0, \quad \frac{d\omega_z}{dt} = \frac{d\omega}{dt}.$$

The theorem is likewise obvious, *à priori.*

The theorems contained in the Examples given above are taken from Schell's *Theorie der Bewegung und der Kräfte*, to which the student is referred for more extended investigations on the subject.

21. A body is rotating round a fixed point O. If OX, OY, OZ be rectangular axes fixed in space, and OA, OB, OC rectangular axes fixed in the body; and if the direction cosines of the latter referred to the former be, respectively, a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3: show that

$$\frac{da_1}{dt} = b_1\omega_3 - c_1\omega_2, \qquad \frac{db_1}{dt} = c_1\omega_1 - a_1\omega_3, \qquad \frac{dc_1}{dt} = a_1\omega_2 - b_1\omega_1,$$

$$\frac{da_2}{dt} = b_2\omega_3 - c_2\omega_2, \qquad \frac{db_2}{dt} = c_2\omega_1 - a_2\omega_3, \qquad \frac{dc_2}{dt} = a_2\omega_2 - b_2\omega_1,$$

$$\frac{da_3}{dt} = b_3\omega_3 - c_3\omega_2, \qquad \frac{db_3}{dt} = c_3\omega_1 - a_3\omega_3, \qquad \frac{dc_3}{dt} = a_3\omega_2 - b_3\omega_1,$$

where $\omega_1, \omega_2, \omega_3$ are the angular velocities of the body round OA, OB, OC.

To form these equations we may suppose velocities of rotation $-\omega_1, -\omega_2, -\omega_3$ round OA, OB, OC, applied to these axes, as well as to OX, OY, OZ; the relative motion is then undisturbed. Take a point on OX, at a distance R from the origin, and call its coordinates referred to OA, OB, OC, ξ, η, ζ. OA, OB, and OC are in consequence of the applied velocities fixed in space, whereas OX is moving.

$$\xi = Ra_1, \quad \eta = Rb_1, \quad \zeta = Rc_1,$$

$$R\frac{da_1}{dt} = \frac{d\xi}{dt} = \eta\,\omega_3 - \zeta\,\omega_2 = R\,(b_1\,\omega_3 - c_1\,\omega_2)\,;$$

whence
$$\frac{da_1}{dt} = b_1\,\omega_3 - c_1\,\omega_2\,;$$

and the others are obtained in a similar manner.

22. A right circular cone is rolling on another fixed in space, the two cones having a common vertex. Given the velocity of rotation of the rolling cone, determine the velocity with which the plane passing through the instantaneous axis turns round the axis of the fixed cone.

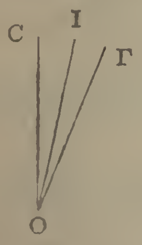
The normal plane through the instantaneous axis contains the axes of both cones. Hence the angle between the two axes remains invariable: and a point on the axis of the rolling cone describes a circle having its centre on the axis of the fixed cone, and its plane perpendicular thereto. It is also at any instant rotating round the instantaneous axis. If we equate the two expressions for the velocity of this point, we get

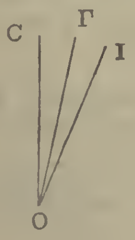
$$\omega \sin \Gamma = \Omega \sin (C + \Gamma), \tag{1}$$

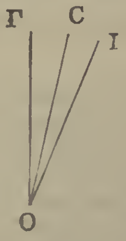
or
$$-\,\omega \sin \Gamma = \Omega \sin (C - \Gamma), \tag{2}$$

or
$$\omega \sin \Gamma = \Omega \sin (\Gamma - C), \tag{3}$$

where ω is the angular velocity of the rolling cone round the instantaneous axis; Ω the angular velocity of the plane containing the instantaneous axis round the axis of the fixed cone; C and Γ the semi-angles of the fixed and moving cones.







The first, second, or third formula is to be used, according as—(1) the cones are *outside* one another, having convex surfaces in contact; (2) the rolling cone is a small cone rolling *inside* a larger one; (3) the rolling cone is the larger cone, and rolls outside a smaller fixed cone, which it contains within it.

In each of these figures OC is the axis of the fixed cone; $O\Gamma$ of the rolling cone; and OI the instantaneous axis. If the angles are supposed to contain their signs implicitly, each being measured from OI, the last formula contains the other two.

23. A body is moving round a fixed point. The motion of the instantaneous axis in the body being completely given, determine its motion in space.

Describe a sphere of radius a round the fixed point; the cone C fixed in space and the cone Γ fixed in the body trace out curves on this sphere, and the motion is accomplished by the one curve rolling on the other. The osculating circle of each of these curves, as it passes through three points on the surface of the sphere, will be a circle of the sphere; and the rolling at any instant will be the same as if one of these circles rolled on the other, or as if the right cone on the osculating circle of Γ as base rolled on the right cone having the osculating circle of C as base. Let r be the radius of curvature of the curve C; ρ of the curve Γ; then

$$\sin C = \frac{r}{a}, \qquad \sin \Gamma = \frac{\rho}{a}, \quad \text{and therefore (Ex. 22),}$$

$$\omega\frac{\rho}{a} = \Omega \sin\left\{\sin^{-1}\frac{\rho}{a} + \sin^{-1}\frac{r}{a}\right\}.$$

Now, if s be the arc of the curve C, and $\frac{ds}{dt}$ the velocity of the point of contact of Γ along it,

$$\Omega = \frac{1}{r}\,\frac{ds}{dt} = \frac{1}{r}\,\frac{d\sigma}{dt},$$

where σ is the arc of Γ; whence

$$\omega\frac{\rho}{a} = \frac{1}{r}\,\frac{d\sigma}{dt}\sin\left\{\sin^{-1}\frac{\rho}{a} + \sin^{-1}\frac{r}{a}\right\}.$$

From this equation r can be determined in terms of t (in terms of which ρ, ω, and $\frac{d\sigma}{dt}$ are supposed to be expressed); and, as $\frac{ds}{dt} = \frac{d\sigma}{dt}$, by eliminating t an equation is obtained between r and s, which is the equation of the curve C; therefore, &c.

24. A body is rotating about a fixed point. Express the element of the curve described by the instantaneous axis on a sphere fixed in the body in terms of the angular velocities round the body-axes.

Let the instantaneous axis at any time make angles λ, μ, ν with the body-axes; let the spherical surface be intersected by the two consecutive positions of the instantaneous axis in I and I'; let $OI = \lambda\omega$: take $O'' = \lambda\omega' = \lambda(\omega + d\omega)$. The projections of II'' on the body-axes are

$$\lambda\, d\omega_1, \quad \lambda\, d\omega_2, \quad \lambda\, d\omega_3,$$

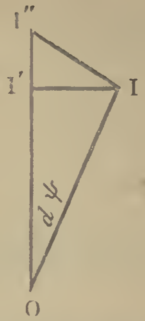
whence $$II''^2 = \lambda^2 (d\omega_1^2 + d\omega_2^2 + d\omega_3^2);$$

but $$II''^2 = I'I''^2 + OI^2\, d\psi^2 = \lambda^2 (d\omega^2 + \omega^2\, d\psi^2).$$

Hence $$d\psi^2 = \frac{1}{\omega^2} (d\omega_1^2 + d\omega_2^2 + d\omega_3^2 - d\omega^2).$$

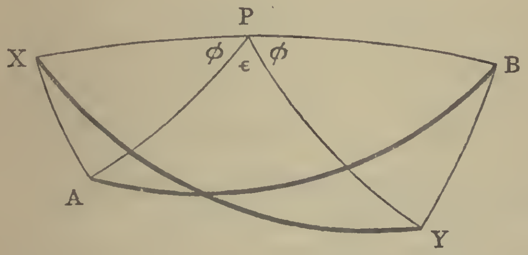
If a be the radius of the sphere, and σ the arc of the curve, we have, therefore,

$$\left(\frac{d\sigma}{dt}\right)^2 = a^2 \left(\frac{d\psi}{dt}\right)^2 = \frac{a^2}{\omega^2} \left\{ \left(\frac{d\omega_1}{dt}\right)^2 + \left(\frac{d\omega_2}{dt}\right)^2 + \left(\frac{d\omega_3}{dt}\right)^2 - \left(\frac{d\omega}{dt}\right)^2 \right\}.$$



25. A body is moving round a fixed point. Being given the axis, a rotation round which brings the body from one position to another, and the magnitude of the rotation, determine the angles which body-axes make in the second position with the space-axes which in the first position coincide with them.

Describe a sphere round the fixed point O. Let two of the space-axes meet it in the points X, Y; and the corresponding body-axes in the points A, B, when the body is in its second position; let P be the pole of rotation; then



$$XPA = YPB = \phi,$$

where ϕ is the given rotation. Let α, β, γ be the angles which the given axis of rotation makes with OX, OY, OZ; and let

$$\cos\alpha = l, \quad \cos\beta = m, \quad \cos\gamma = n;$$

then $$XP = AP = \alpha, \quad YP = BP = \beta.$$

Let angle $APY = \epsilon$; then

$$\cos XA = \cos\phi \sin^2\alpha + \cos^2\alpha = l^2 + (1 - l^2)\cos\phi,$$

$$\cos YA = \cos\epsilon \sin\alpha \sin\beta + \cos\alpha \cos\beta.$$

Now $$\cos\frac{\pi}{2} = \cos AB = \cos(\phi + \epsilon) \sin\alpha \sin\beta + \cos\alpha \cos\beta,$$

whence $$\cos(\phi + \epsilon) = -\cot\alpha \cot\beta;$$

therefore $$\sin(\phi + \epsilon) = \frac{\sqrt{1 - \cos^2 \alpha - \cos^2 \beta}}{\sin \alpha \sin \beta} = \frac{n}{\sin \alpha \sin \beta}.$$

Hence $$\sin \alpha \sin \beta \cos \epsilon = n \sin \phi - \cos \alpha \cos \beta \cos \phi,$$

and, therefore, $$\cos YA = lm(1 - \cos \phi) + n \sin \phi.$$

Again $$\cos XB = \cos(2\phi + \epsilon) \sin \alpha \sin \beta + \cos \alpha \cos \beta.$$

Expanding $\cos(2\phi + \epsilon)$, and substituting for $\sin(\phi + \epsilon)$ and $\cos(\phi + \epsilon)$ their values given above,

$$\cos XB = lm(1 - \cos \phi) - n \sin \phi.$$

The values of the cosines of the remaining angles can now be written down from symmetry. If we put

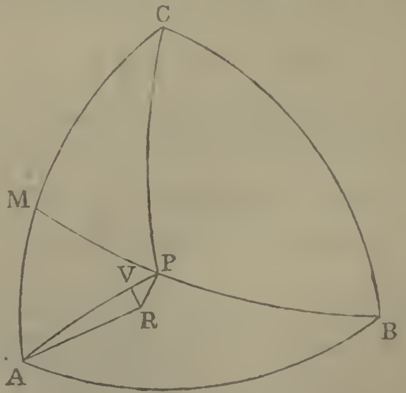
$$\upsilon = \cos \tfrac{1}{2}\phi, \quad \lambda = l \sin \tfrac{1}{2}\phi, \quad \mu = m \sin \tfrac{1}{2}\phi, \quad \nu = n \sin \tfrac{1}{2}\phi,$$

we have the following table of the values of $\cos XA$, &c. :—

	X	Y	Z
A	$\upsilon^2 + \lambda^2 - \mu^2 - \nu^2$	$2(\mu\lambda + \upsilon\nu)$	$2(\nu\lambda - \upsilon\mu)$
B	$2(\lambda\mu - \upsilon\nu)$	$\upsilon^2 + \mu^2 - \nu^2 - \lambda^2$	$2(\nu\mu + \upsilon\lambda)$
C	$2(\lambda\nu + \upsilon\mu)$	$2(\mu\nu - \upsilon\lambda)$	$\upsilon^2 + \nu^2 - \lambda^2 - \mu^2$

The quantities l, m, n, ϕ are called Rodrigues' coordinates. (Thomson and Tait, *Natural Philosophy*, § 95.)

26. The body is rotating round OA with an angular velocity ω_1; determine the differential coefficients of ϕ, l, m, n with respect to the time.



We have to find the magnitude and axis of the rotation which is the resultant of the rotation ϕ round (l, m, n), and of $\omega_1\, dt$ round OA. If this rotation be ϕ', and its direction cosines l', m', n'; $\phi' - \phi = \frac{d\phi}{dt}\, dt$, $l' - l = \frac{dl}{dt}\, dt$, &c.

We proceed, according to the method of Ex. 5. Let A, B, C be the points in which OA, OB, OC meet the sphere described round O, and P the point in which the sphere is met by the line (l, m, n). Make $APR = \frac{1}{2}\phi$, and $PAR = -\frac{1}{2}\omega_1\,dt$. (The positive direction of rotation is supposed to be counter-clockwise). R is the pole of the resultant rotation.

Draw RV at right angles to AP,

$$AR = AP - VP = AP - PR\cos APR.$$

In like manner
$$BR = BP - PR\cos BPR,$$

$$CR = CP - PR\cos CPR;$$

whence, remembering that PR is infinitely small, we have

$$l' = \cos AR = l + PR\sin AP\cos APR,$$

$$m' = \cos BR = m + PR\sin BP\cos BPR,$$

$$n' = \cos CR = n + PR\sin CP\cos CPR,$$

$$PR = \tfrac{1}{2}\omega_1\,dt\,.\,\frac{\sin AR}{\sin\frac{1}{2}\phi} = \tfrac{1}{2}\omega_1\,dt\,.\,\frac{\sin AP}{\sin\frac{1}{2}\phi};$$

therefore
$$2\frac{dl}{dt} = \omega_1\sin^2 AP\,\frac{\cos APR}{\sin\frac{1}{2}\phi} = \omega_1(1-l^2)\cot\tfrac{1}{2}\phi,$$

$$2\frac{dm}{dt} = \omega_1\sin AP\sin BP\,\frac{\cos BPR}{\sin\frac{1}{2}\phi},$$

$$2\frac{dn}{dt} = \omega_1\sin AP\sin CP\,\frac{\cos CPR}{\sin\frac{1}{2}\phi},$$

$$\cos BPR = \cos(BPA - APR),$$

and
$$\cos\frac{\pi}{2} = \cos BPA\sin BP\sin AP + \cos BP\cos AP;$$

whence
$$\cos BPA = -\frac{lm}{\sqrt{\{(1-l^2)(1-m^2)\}}},\quad \sin BPA = \frac{n}{\sqrt{\{(1-l^2)(1-m^2)\}}},$$

and
$$2\frac{dm}{dt} = \omega_1(n - lm\cot\tfrac{1}{2}\phi);$$

and, in like manner, as
$$CPR = CPA + APR,$$

$$2\frac{dn}{dt} = -\omega_1(m + nl\cot\tfrac{1}{2}\phi).$$

Again
$$\phi' = 2(\pi - ARP),$$

and
$$\cos ARP = \frac{\omega_1\,dt}{2}\cos AP\sin\frac{\phi}{2} - \cos\frac{\phi}{2};$$

therefore $$\cos\tfrac{1}{2}\phi' = \cos\tfrac{1}{2}\phi - \tfrac{1}{2}\sin\tfrac{1}{2}\phi\, l\,\omega_1\, dt,$$

whence $$\frac{d\phi}{dt} = l\omega_1.$$

27. A body is moving round a fixed point O, with velocities of rotation ω_1, ω_2, ω_3, round three rectangular axes OA, OB, OC fixed in the body. Determine the differential coefficients of Rodrigues' coordinates with respect to the time.

By means of the last example we can write down the changes produced on ϕ, l, m, n by each of the rotations $\omega_1\, dt$, $\omega_2\, dt$, $\omega_3\, dt$.

Adding, and dividing by dt, we get

$$2\frac{dl}{dt} = -\omega_2 n + \omega_3 m + \cot\tfrac{1}{2}\phi\left\{\omega_1 - l(l\omega_1 + m\omega_2 + n\omega_3)\right\},$$

$$2\frac{dm}{dt} = -\omega_3 l + \omega_1 n + \cot\tfrac{1}{2}\phi\left\{\omega_2 - m\,(l\omega_1 + m\omega_2 + n\omega_3)\right\},$$

$$2\frac{dn}{dt} = -\omega_1 m + \omega_2 l + \cot\tfrac{1}{2}\phi\left\{\omega_3 - n\,(l\omega_1 + m\omega_2 + n\omega_3)\right\},$$

$$\frac{d\phi}{dt} = l\omega_1 + m\omega_2 + n\,\omega_3\,;$$

whence, also, we obtain

$$2\frac{d\upsilon}{dt} = -\omega_1\lambda - \omega_2\mu - \omega_3\nu,$$

$$2\frac{d\lambda}{dt} = \omega_1\upsilon - \omega_2\nu + \omega_3\mu,$$

$$2\frac{d\mu}{dt} = \omega_2\upsilon - \omega_3\lambda + \omega_1\nu,$$

$$2\frac{d\nu}{dt} = \omega_3\upsilon - \omega_1\mu + \omega_2\lambda,$$

where υ, λ, μ, ν have the same meaning as in Ex. 25.

28. If θ, ϕ, ψ, ω_1, ω_2, ω_3 have the same significations as in Art. 238, show that

$$\omega_1 = \sin\phi\frac{d\theta}{dt} - \sin\theta\cos\phi\frac{d\psi}{dt},$$

$$\omega_2 = \cos\phi\frac{d\theta}{dt} + \sin\theta\sin\phi\frac{d\psi}{dt},$$

$$\omega_3 = \frac{d\phi}{dt} + \cos\theta\frac{d\psi}{dt},$$

29. Show also that

$$\cos AX = \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi,$$

$$\cos AY = \cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi,$$

$$\cos AZ = -\sin\theta\cos\phi;$$

$$\cos BX = -\cos\theta\sin\phi\cos\psi - \cos\phi\sin\psi,$$

$$\cos BY = -\cos\theta\sin\phi\sin\psi + \cos\phi\cos\psi,$$

$$\cos BZ = \sin\theta\sin\phi;$$

$$\cos CX = \sin\theta\cos\psi,$$

$$\cos CY = \sin\theta\sin\psi,$$

$$\cos CZ = \cos\theta.$$

These equations follow readily from the figure on p. 298, remembering that

$$SEX = \theta, \quad AS = ACS = \phi, \quad EY = QX = \psi.$$

SECTION II.—*Kinetics.*

241. **Equations of Motion of a Body having a Fixed Point.**—As the body has a fixed point, D'Alembert's Principle requires that the moments of the forces of inertia round three axes through the point should be equal and opposite to the moments of the applied forces. If we take for axes three lines fixed in space, intersecting at right angles at the point, we have, Art. 203,

$$L = \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \Sigma m\frac{d}{dt}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right),$$

with two other equations of like form.

Now, the body having a fixed point must (Art. 229) be rotating round some axis through it; and if ω_x, ω_y, ω_z be the angular velocities round the axes of coordinates, we have, substituting for $\frac{dy}{dt}$ and $\frac{dz}{dt}$ their values from Art. 235,

$$\left.\begin{aligned}\Sigma m \frac{d}{dt}\{(y^2+z^2)\,\omega_x - xy\omega_y - xz\omega_z\} &= L\\ \Sigma m \frac{d}{dt}\{(z^2+x^2)\,\omega_y - yz\omega_z - yx\omega_x\} &= M\\ \Sigma m \frac{d}{dt}\{(x^2+y^2)\,\omega_z - zx\omega_x - zy\omega_y\} &= N\end{aligned}\right\}. \quad (1)$$

Performing the differentiations, substituting for $\frac{dx}{dt}$, &c., their values from equations (4), Art. 235, and, finally, multiplying by the element of mass, and integrating through the body, we obtain the complete general value of

$$\Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right).$$

This result is seldom of much use, as the definite integrals which it contains are in general functions of the time.

If, however, we suppose the axes fixed in space to coincide with the instantaneous position of the principal axes of the body passing through the origin, the result becomes much simplified, and is readily obtained, as follows :—

Let the coordinates of any point of the body referred to its principal axes be ξ, η, ζ, and let the angular velocities round these axes be ω_1, ω_2, ω_3; then

$$x = \xi,\ y = \eta,\ z = \zeta,\ \omega_x = \omega_1,\ \omega_y = \omega_2,\ \omega_z = \omega_3,\ \frac{dx}{dt} = -\eta\omega_3 + \zeta\omega_2,$$

with similar expressions for $\frac{dy}{dt}$ and $\frac{dz}{dt}$; and, by Art. 236,

$$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt},\quad \frac{d\omega_y}{dt} = \frac{d\omega_2}{dt},\quad \frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}.$$

Also, as the axes are the principal axes at the origin,

$$\Sigma m\eta\zeta = \Sigma m\zeta\xi = \Sigma m\xi\eta = 0.$$

If we observe that $\frac{dx}{dt}$ does not contain ξ, we see that $\Sigma mx\frac{dx}{dt}$, and the two similar expressions must vanish.

Again, $$\Sigma m\left(x\frac{dy}{dt}+y\frac{dx}{dt}\right)=\Sigma m\,(\xi^2-\eta^2)\,\omega_3,$$

and $$\Sigma m\left(x\frac{dz}{dt}+z\frac{dx}{dt}\right)=\Sigma m\,(\zeta^2-\xi^2)\,\omega_2.$$

Hence $$\Sigma m\,\frac{d}{dt}\{(y^2+z^2)\,\omega_x-xy\,\omega_y-xz\,\omega_z\}=\frac{d\omega_1}{dt}\,\Sigma m\,(\eta^2+\zeta^2)$$
$$-\,\omega_2\,\omega_3\,\Sigma m\,(\zeta^2-\eta^2).$$

If A, B, C be the three principal moments of inertia of the body at the origin,

$$\Sigma m\,(\eta^2+\zeta^2)=A,\quad \Sigma m\,(\zeta^2-\eta^2)=B-C.$$

Hence, finally,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\,\omega_3&=L\\ B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\,\omega_1&=M\\ C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\,\omega_2&=N\end{aligned}\right\},\qquad (2)$$

where L, M, N are the moments of the applied forces round the three principal axes passing through the origin which are fixed in the body but not in space.

The equations above are due to Euler and are called by his name.

242. **Of the Terms in Euler's Equations.**—The change produced in ω_1 in the element of time is given by the equation

$$Ad\omega_1=(B-C)\,\omega_2\,\omega_3\,dt+L\,dt.$$

The first term on the right-hand side of this equation results from the angular velocities already existing round the axes of y and z. In consequence of these velocities each point of the body, in virtue of its connexions with the other points, exerts a force on the entire body. If we consider the centrifugal forces resulting from the motion of the body, we can determine the moments L', M', N' of these forces round the axes, as follows:—

Let α, β, γ be the angles which the instantaneous axis of rotation makes with the axes of coordinates fixed in space; p the perpendicular distance from this axis to any point x, y, z of the body; q the intercept between the origin and the foot of p; r the radius vector to the point xyz; and ω the angular velocity of the body round the instantaneous axis. The centrifugal force at the point x, y, z is $mp\omega^2$ acting along p; and the component of this force along the axis of x is $m\omega^2$ multiplied by the projection of p.

If we project the triangle formed by rpq on the axis of x, we have

projection of p = projection of r − projection of $q = x - q \cos \alpha$,

and $$q = x \cos \alpha + y \cos \beta + z \cos \gamma;$$

whence force along axis of x

$$= m\omega^2 \{x - (x \cos \alpha + y \cos \beta + z \cos \gamma) \cos \alpha\}$$

$$= m\omega^2 \{x (\cos^2 \beta + \cos^2 \gamma) - y \cos \alpha \cos \beta - z \cos \alpha \cos \gamma\}$$

$$= m \{x (\omega_y^2 + \omega_z^2) - y\, \omega_x \omega_y - z\, \omega_x \omega_z\},$$

remembering that

$$\cos \alpha = \frac{\omega_x}{\omega}, \quad \cos \beta = \frac{\omega_y}{\omega}, \quad \cos \gamma = \frac{\omega_z}{\omega}.$$

In like manner for the force along the axis of y, we have

$$m \{y (\omega_z^2 + \omega_x^2) - z\, \omega_y \omega_z - x\, \omega_y \omega_x\},$$

and for that along the axis of z,

$$m \{z (\omega_x^2 + \omega_y^2) - x\, \omega_z \omega_x - y\, \omega_z \omega_y\};$$

whence, taking moments round the axis of x, and integrating through the entire body, we obtain

$$L' = (\omega_y^2 - \omega_z^2) \int yz\, dm + \omega_y \omega_z \int (z^2 - y^2)\, dm - \omega_z \omega_x \int xy\, dm$$
$$+ \omega_x \omega_y \int xz\, dm.$$

If we now suppose the axes to coincide with the principal axes of the body, every term in L' vanishes except $\omega_y \omega_z \int (z^2 - y^2)\, dm$, and we get

$$L' = (B - C)\, \omega_2 \omega_3. \qquad (3)$$

Accordingly, the couple whose components round the three axes are $(B - C)\, \omega_2 \omega_3$, &c. is called the *centrifugal couple.*

243. **Euler's Equations for Impulses.**—If a body having a fixed point be acted on by impulsive forces, and $\underset{\cdot}{L}$, $\underset{\cdot}{M}$, $\underset{\cdot}{N}$ be the moments of their impulses round the principal axes through the point, we have, from equations (4) Art. 203,

$$A(\omega_1 - \omega'_1) = \underset{\cdot}{L}, \quad B(\omega_2 - \omega'_2) = \underset{\cdot}{M}, \quad C(\omega_3 - \omega'_3) = \underset{\cdot}{N}. \qquad (4)$$

In fact, $\Sigma m\,(y\dot{z} - z\dot{y}) = A(\omega_1 - \omega_1')$ when the axes fixed in space coincide with the instantaneous positions of the principal axes through the fixed point, ω_1 being the angular velocity of the body round the axis of ξ, immediately after the impulse, and ω_1' its angular velocity round the same axis, immediately before.

Equations (4) can also be deduced easily from the corresponding equations for continuous forces.

244. **Couple of Principal Moments.**—If a body be moving in any manner round a fixed point, we may imagine its actual velocity at any instant to be produced by an impulsive couple which acted on it at the previous instant. By the last Article the components of this couple round the principal axes of the body are $A\omega_1$, $B\omega_2$, $C\omega_3$, and the axis of the couple is called the *Axis of Principal Moments.*

If a tangent plane be drawn at the point of intersection of the instantaneous axis of rotation with the momental ellipsoid corresponding to the fixed point round which the body is rotating, the perpendicular from the centre on this tangent plane is the Axis of Principal Moments. This is obvious, when we remember that the direction cosines of this axis are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$; and those of the instantaneous axis of rotation to ω_1, ω_2, ω_3; and that the equation of the momental ellipsoid is

$$Ax^2 + By^2 + Cz^2 = K.$$

The axis of the centrifugal couple is at right angles to the axis of principal moments, and to the axis of rotation.

The direction cosines of the axis of the centrifugal couple are proportional to

$$(B - C)\,\omega_2\,\omega_3, \quad (C - A)\,\omega_3\,\omega_1, \quad (A - B)\,\omega_1\,\omega_2;$$

whence it is seen at once that the conditions for its being perpendicular to the two other lines are fulfilled.

If a central section of the momental ellipsoid be taken through the instantaneous axis of rotation and the axis of the centrifugal couple, these two lines coincide with the principal axes of this section.

The lines in question are at right angles, and one is parallel to the tangent plane through the point where the other intersects the ellipsoid.

245. **Vis Viva of a Body having a Fixed Point.**—The equation of vis viva is in this case immediately deducible from Art. 130. As the body has a fixed point, it is at any instant rotating round some axis through it; whence, I being the moment of inertia round the instantaneous axis, the vis viva is $I\omega^2$.

Again, since $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$ are the direction cosines of the axis of rotation referred to the principal axes through the fixed point,

$$I = A\left(\frac{\omega_1}{\omega}\right)^2 + B\left(\frac{\omega_2}{\omega}\right)^2 + C\left(\frac{\omega_3}{\omega}\right)^2; \quad (\textit{Int. Calc.}, \text{Art. } 215),$$

whence the equation of vis viva becomes

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2\Sigma\int(Xdx + Ydy + Zdz) + \text{const.} \quad (5)$$

246. **Motion consisting of Successive Rotations.**—If the whole motion of a body consist of successive rotations (not necessarily executed round lines passing through the same point), the vis viva at any time is $I\omega^2$, where I is the moment of inertia, and ω the angular velocity round the instantaneous axis; hence

$$I\omega^2 = 2\Sigma\int (Xdx + Ydy + Zdz) + \text{const.};$$

therefore $$\frac{d}{dt}(I\omega^2) = 2\Sigma\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right).$$

Let p be the perpendicular on the instantaneous axis from the point x, y, z; and let the direction of the motion of this point make angles α, β, γ with the axes; then

$$\frac{dx}{dt} = p\omega\cos\alpha, \quad \frac{dy}{dt} = p\omega\cos\beta, \quad \frac{dz}{dt} = p\omega\cos\gamma;$$

whence $$\frac{d}{dt}(I\omega^2) = 2\omega\Sigma p\,(X\cos\alpha + Y\cos\beta + Z\cos\gamma);$$

but $$(X\cos\alpha + Y\cos\beta + Z\cos\gamma)\,p$$

is the moment of the force applied at the point x, y, z round the instantaneous axis. Hence, if J be the moment of the entire system of applied forces round the instantaneous axis,

$$\frac{d}{dt}(I\omega^2) = 2\omega J,$$

or $$\frac{1}{\omega}\frac{d}{dt}(I\omega^2) = 2J. \qquad (6)$$

If the body be such that the moments of inertia round the different instantaneous axes are equal, this equation takes the simple form

$$I\frac{d\omega}{dt} = J. \qquad (7)$$

If the body have a fixed point, equation (6) can be deduced at once from Euler's Equations. For if we multiply the first of these by ω_1; the second by ω_2; the third by ω_3, and add, we have

$$A\omega_1 \frac{d\omega_1}{dt} + B\omega_2 \frac{d\omega_2}{dt} + C\omega_3 \frac{d\omega_3}{dt} = L\omega_1 + M\omega_2 + N\omega_3 ;$$

but
$$2\left(A\omega_1 \frac{d\omega_1}{dt} + B\omega_2 \frac{d\omega_2}{dt} + C\omega_3 \frac{d\omega_3}{dt}\right) = \frac{d}{dt}(I\omega^2) ;$$

and, as $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$ are the direction cosines of the instantaneous axis,

$$L\frac{\omega_1}{\omega} + M\frac{\omega_2}{\omega} + N\frac{\omega_3}{\omega} = J ;$$

whence
$$\frac{1}{\omega}\frac{d}{dt}(I\omega^2) = 2J.$$

247. **Stress Exerted by a Body on a Fixed Point.**—In order to determine the force exerted by a fixed point on a body, we have only to consider the point as replaced by a force, whose components are X', Y', Z' passing through it. We may then consider the body as free, and we have, by Article 204,

$$\mathfrak{M}\frac{d^2\bar{x}}{dt^2} = \Sigma X + X',$$

with two similar equations.

But as the body is rotating round the origin, if we suppose the axes fixed in space to coincide at the instant under consideration with the principal axes through the origin, we have

$$\frac{d^2\bar{x}}{dt^2} = -\bar{y}\frac{d\omega_3}{dt} + \bar{z}\frac{d\omega_2}{dt} + \omega_1(\bar{y}\omega_2 + \bar{z}\omega_3) - (\omega_2^2 + \omega_3^2)\bar{x}.$$

Substituting for $\frac{d\omega_3}{dt}$ and $\frac{d\omega_2}{dt}$ from Euler's Equations, we get, finally,

$$\frac{d^2\bar{x}}{dt^2} = -\bar{y}\frac{N}{C} + \bar{z}\frac{M}{B} + \omega_1(B + C - A)\left(y\frac{\omega_2}{C} + \bar{z}\frac{\omega_3}{B}\right) - (\omega_2^2 + \omega_3^2)\bar{x}.$$

If S_1, S_2, S_3 are the components of the stress on the fixed point at any time, in the directions occupied at the instant by the principal axes of the body, $S_1 = -X'$, and therefore

$$\left.\begin{aligned}
S_1 &= \Sigma X - \mathfrak{M}\left\{-\bar{\eta}\frac{N}{C} + \bar{\zeta}\frac{M}{B} + \omega_1(B+C-A)\left(\bar{\eta}\frac{\omega_2}{C} + \bar{\zeta}\frac{\omega_3}{B}\right) - (\omega_2^2+\omega_3^2)\,\bar{\xi}\right\}\\
S_2 &= \Sigma Y - \mathfrak{M}\left\{-\bar{\zeta}\frac{L}{A} + \bar{\xi}\frac{N}{C} + \omega_2(C+A-B)\left(\bar{\zeta}\frac{\omega_3}{A} + \bar{\xi}\frac{\omega_1}{C}\right) - (\omega_3^2+\omega_1^2)\,\bar{\eta}\right\}\\
S_3 &= \Sigma Z - \mathfrak{M}\left\{-\bar{\xi}\frac{M}{B} + \bar{\eta}\frac{L}{A} + \omega_3(A+B-C)\left(\bar{\xi}\frac{\omega_1}{B} + \bar{\eta}\frac{\omega_2}{A}\right) - (\omega_1^2+\omega_2^2)\,\bar{\zeta}\right\}
\end{aligned}\right\}, \quad (8)$$

where $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ are the coordinates of the centre of inertia referred to the principal axes through the fixed point, and are absolute constants; ΣX is the sum of the components of the applied forces parallel to one of these axes, and L the moment round it of the same forces. ΣX, ΣY, ΣZ, L, M, N are in general variable with the time.

248. **Impulses Exerted by a Body on a Fixed Point.**—If $\underset{\cdot}{S}_1$, $\underset{\cdot}{S}_2$, $\underset{\cdot}{S}_3$ are the impulses of the instantaneous stresses exerted by a body on a fixed point, in consequence of the action on the body of any system of impulses, we obtain, in a similar manner, by Arts. 204, 235, and 243,

$$\left.\begin{aligned}
\underset{\cdot}{S}_1 &= \Sigma \underset{\cdot}{X} + \mathfrak{M}\left\{\bar{\eta}\,\frac{\underset{\cdot}{N}}{C} - \bar{\zeta}\,\frac{\underset{\cdot}{M}}{B}\right\}\\
\underset{\cdot}{S}_2 &= \Sigma \underset{\cdot}{Y} + \mathfrak{M}\left\{\bar{\zeta}\,\frac{\underset{\cdot}{L}}{A} - \bar{\xi}\,\frac{\underset{\cdot}{N}}{C}\right\}\\
\underset{\cdot}{S}_3 &= \Sigma \underset{\cdot}{Z} + \mathfrak{M}\left\{\bar{\xi}\,\frac{\underset{\cdot}{M}}{B} - \bar{\eta}\,\frac{\underset{\cdot}{L}}{A}\right\}
\end{aligned}\right\}. \quad (9)$$

249. **Motion of a Free Body.**—In Art. 205 it was shown that if ξ, η, ζ be the coordinates of any point of a body referred to axes drawn at the centre of inertia parallel to fixed directions in space, the equations

$$\Sigma m\left(\xi\frac{d^2\eta}{dt^2} - \eta\frac{d^2\xi}{dt^2}\right) = \Sigma\,(\xi Y - \eta X),$$

$$\Sigma m\left(\eta\frac{d^2\zeta}{dt^2}-\zeta\frac{d^2\eta}{dt^2}\right)=\Sigma\,(\eta Z-\zeta Y),$$

$$\Sigma m\left(\zeta\frac{d^2\xi}{dt^2}-\xi\frac{d^2\zeta}{dt^2}\right)=\Sigma\,(\zeta X-\xi Z)$$

hold good.

As the coordinates ξ, η, ζ are relative to a point fixed in the body, their differential coefficients can be expressed in terms of the angular velocities of the body round this point. Hence these equations can be transformed in a manner similar to that employed in the deduction of Euler's Equations. Thus we obtain the equations—

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}-(B-C)\,\omega_2\,\omega_3&=L\\ B\frac{d\omega_2}{dt}-(C-A)\,\omega_3\,\omega_1&=M\\ C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\,\omega_2&=N \end{aligned}\right\},\qquad(10)$$

where ω_1, ω_2, ω_3 are the angular velocities; A, B, C the moments of inertia; and L, M, N the moments of the applied forces round the three principal axes of the body at the centre of inertia.

If we call the mass of the body $\mathfrak{M}$, we have also the three equations

$$\mathfrak{M}\frac{d^2\bar{x}}{dt^2}=\Sigma X,\qquad \mathfrak{M}\frac{d^2\bar{y}}{dt^2}=\Sigma Y,\qquad \mathfrak{M}\frac{d^2\bar{z}}{dt^2}=\Sigma Z,\quad(11)$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the centre of inertia referred to any three rectangular axes fixed in space; and ΣX, ΣY, ΣZ are the sums of the components of the applied forces parallel to these axes.

If the body be subject to geometrical conditions, these conditions must in general be replaced by indeterminate reactions before equations (11) can be employed, and these reactions must appear among the applied forces. This is obvious from considering the statical problem to which, by D'Alembert's Principle, the dynamical problem is reduced.

The corresponding equations for impulses can be at once written down: they are

$$\left.\begin{aligned} &A(\omega_1 - \omega'_1) = \underset{\cdot}{L},\ B(\omega_2 - \omega'_2) = \underset{\cdot}{M},\ C(\omega_3 - \omega'_3) = \underset{\cdot}{N} \\ &\mathfrak{M}\dot{\bar{x}} = \Sigma \underset{\cdot}{X},\quad \mathfrak{M}\dot{\bar{y}} = \Sigma \underset{\cdot}{Y},\quad \mathfrak{M}\dot{\bar{z}} = \Sigma \underset{\cdot}{Z} \end{aligned}\right\}. \quad (12)$$

If the motion of the body be constrained in any way, the conditions of restraint can, as in the former case, be replaced by indeterminate reactions.

From equations (10), (11), (12) it is obvious that a force or impulse whose direction passes through the centre of inertia of a free body produces a motion of translation only, whereas a force not passing through the centre of inertia produces both a translation and a rotation.

250. **Equations of Motion referred to Body-Axes.**—If u, v, w be the components of the velocity of the centre of inertia at any instant in the directions of the principal axes of the body at that point, the accelerations of the centre of inertia in these directions are, by Art. 237,

$$\frac{du}{dt} - v\omega_3 + w\omega_2, \text{ \&c.};$$

whence, if ΣX, ΣY, ΣZ be the sums of the components of the applied forces at any instant, parallel to the principal axes through the centre of inertia, its equations of motion are

$$\left.\begin{aligned} \mathfrak{M}\left\{\frac{du}{dt} - v\omega_3 + w\omega_2\right\} &= \Sigma X \\ \mathfrak{M}\left\{\frac{dv}{dt} - w\omega_1 + u\omega_3\right\} &= \Sigma Y \\ \mathfrak{M}\left\{\frac{dw}{dt} - u\omega_2 + v\omega_1\right\} &= \Sigma Z \end{aligned}\right\}; \qquad (13)$$

and the corresponding equations for impulses are

$$\mathfrak{M}(u - u') = \Sigma \underset{\cdot}{X},\ \mathfrak{M}(v - v') = \Sigma \underset{\cdot}{Y},\ \mathfrak{M}(w - w') = \Sigma \underset{\cdot}{Z}. \quad (14)$$

251. **Motion of a Free Body relative to its Centre of Inertia.**—As the equations for determining the motion

of a body relative to its centre of inertia are the same as if the centre of inertia were a fixed point, the theorems of Art. 244, in reference to the instantaneous axis of rotation, the axis of the centrifugal couple, and the axis of principal moments, hold good.

252. **General Expression for the Vis Viva of a Body.**—As the motion of a body relative to one of its points must always consist of a rotation round some axis through the point, it follows, from Art. 131, that if a body be free,

$$\Sigma mv^2 = \mathfrak{M} V^2 + I\omega^2,$$

where $\mathfrak{M}$ is the mass of the body; V the velocity of its centre of inertia; I the moment of inertia, and ω the angular velocity, round the instantaneous axis through the centre of inertia.

As was shown in Art. 245,

$$I\omega^2 = A\omega_1^2 + B\omega_2^2 + C\omega_3^2.$$

Again, if a, b, c, i, j, k be the moments and products of inertia round three axes through the centre of inertia, at right angles to each other, and parallel to fixed directions in space, and ω_x, ω_y, ω_z the angular velocities of the body round these axes,

$$I\omega^2 = a\omega_x^2 + b\omega_y^2 + c\omega_z^2 - 2i\omega_y\,\omega_z - 2j\omega_z\,\omega_x - 2k\omega_x\,\omega_y\,;$$

whence we have the two expressions

$$\Sigma mv^2 = \mathfrak{M} V^2 + A\omega_1^2 + B\omega_2^2 + C\omega_3^2, \tag{15}$$

$$\Sigma mv^2 = \mathfrak{M} V^2 + a\omega_x^2 + b\omega_y^2 + c\omega_z^2 - 2i\omega_y\,\omega_z - 2j\omega_z\,\omega_x - 2k\omega_x\,\omega_y. \tag{16}$$

253. **Moments of Momentum.**—If x, y, z be rectangular axes parallel to directions fixed in space, the quantities

$$\Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right),\ \Sigma m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right),\ \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right),$$

are at any instant the moments of the momenta of the different elements of the body round the axes.

If we call these moments H_1, H_2, H_3, they are equivalent to a single moment H round an axis whose direction cosines are $\frac{H_1}{H}$, $\frac{H_2}{H}$, $\frac{H_3}{H}$; H being given by the equation

$$H^2 = H_1^2 + H_2^2 + H_3^2.$$

This line is called the *momentum axis* relative to the origin. As it is the axis of the couple which is the resultant of the moments of the momenta of the different elements of the body, it is plain that its direction is independent of the directions of the axes.

If Sdt be twice the sum of the projections of the elementary areas described by all the points of the body round the origin, each multiplied by the corresponding element of mass, on a plane whose normal makes an angle θ with the momentum axis,

$$S = H \cos \theta. \tag{17}$$

This may be proved in the following manner:—Let hdt be double the elementary area described by the element whose mass is m round the origin; and let a, β, γ be the cosines of the angles its plane makes with the coordinate planes; then, λ, μ, ν being the direction cosines of the normal to the plane of S,

$$S = \Sigma mh\,(a\lambda + \beta\mu + \gamma\nu) = \lambda\Sigma mha + \mu\Sigma mh\beta + \nu\Sigma mh\gamma$$

$$= \lambda H_1 + \mu H_2 + \nu H_3 = H\left\{\lambda\frac{H_1}{H} + \mu\frac{H_2}{H} + \nu\frac{H_3}{H}\right\} = H\cos\theta.$$

Hence, *the multiple sum of the projections of the elementary areas on the plane at right angles to the momentum axis is a maximum.*

This plane is called the *Principal Plane* relative to the origin. From what has been just proved, we see again that its position is independent of the directions of the axes.

If ξ, η, ζ be a second set of rectangular axes through the origin parallel to directions fixed in space, and if the direction cosines of ξ, referred to x, y, z, be a_1, a_2, a_3; of η, b_1, b_2, b_3; of ζ, c_1, c_2, c_3; we have, as particular cases of what has been proved above,

$$\left.\begin{aligned} \Sigma m\left(\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt}\right) &= H_1 a_1 + H_2 a_2 + H_3 a_3 \\ \Sigma m\left(\zeta\frac{d\xi}{dt} - \xi\frac{d\zeta}{dt}\right) &= H_1 b_1 + H_2 b_2 + H_3 b_3 \\ \Sigma m\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) &= H_1 c_1 + H_2 c_2 + H_3 c_3 \end{aligned}\right\}. \qquad (18)$$

The preceding theorems of this Article are true for any system of moving points, and whether the origin be fixed or movable.

In the case of a body moving round a fixed point, if three lines coinciding with the instantaneous positions of the principal axes through the point be taken as axes, we have, by equations (4), Art. 235,

$$H_1 = A\omega_1, \quad H_2 = B\omega_2, \quad H_3 = C\omega_3.$$

Hence, *if a body be rotating round a fixed point, its moment of momentum H round this point is given by the equation*

$$H^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2. \qquad (19)$$

Again, if x, y, z be three rectangular axes fixed in space passing through the fixed point of the body, and a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 be the direction cosines of the principal axes of the body referred to the axes fixed in space,

$$\left.\begin{aligned} \Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) &= A\omega_1 a_1 + B\omega_2 a_2 + C\omega_3 a_3 \\ \Sigma m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) &= A\omega_1 b_1 + B\omega_2 b_2 + C\omega_3 b_3 \\ \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) &= A\omega_1 c_1 + B\omega_2 c_2 + C\omega_3 c_3 \end{aligned}\right\}. \qquad (20)$$

As the direction cosines of the momentum axis, relative to the principal axes through the fixed point, are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$, the momentum axis coincides with the axis of Principal Moments. This is also immediately obvious from D'Alembert's Principle applied to impulses.

If the body be free, these theorems still hold good for the motion relative to any point of the body.

The moment of momentum round an axis through any point O is equal to the corresponding quantity with respect to a parallel axis through the centre of inertia, together with the moment of momentum relative to O of the whole mass concentrated at the centre of inertia and moving with it.

For let x, y, z be the coordinates of any point, referred to axes whose directions are fixed in space and which meet in O, and let ξ, η, ζ be those of the same point, referred to parallel axes at the centre of inertia; then

$$\Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = \Sigma m\left\{(\bar{y}+\eta)\frac{d(\bar{z}+\zeta)}{dt} - (\bar{z}+\zeta)\frac{d(\bar{y}+\eta)}{dt}\right\}$$

$$= (\Sigma m)\left(\bar{y}\frac{d\bar{z}}{dt} - \bar{z}\frac{d\bar{y}}{dt}\right) + \Sigma m\left(\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt}\right),$$

since $\Sigma m\,\eta = \Sigma m\,\zeta = 0, \quad \Sigma m\frac{d\eta}{dt} = \Sigma m\frac{d\zeta}{dt} = 0,$

or $$\Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = \mathfrak{M}\left(\bar{y}\frac{d\bar{z}}{dt} - \bar{z}\frac{d\bar{y}}{dt}\right) + \Sigma m\left(\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt}\right).$$

254. **Relations between Forces and Moments of Momentum.**—If H_1, H_2, H_3 be the moments of momentum of a body round three rectangular axes fixed in space intersecting at O, and L, M, N the moments of the applied forces round the same axes; then by Art. 241,

$$\frac{dH_1}{dt} = L, \quad \frac{dH_2}{dt} = M, \quad \frac{dH_3}{dt} = N. \qquad (21)$$

If O be the centre of inertia, and the directions of the axes be fixed in space, these equations hold good, whether O be fixed or moving (Art. 205).

If one point O of the body be fixed in space, and a, b, c, i, j, k be the moments and products of inertia at any instant round axes fixed in space intersecting at O, the equations above may be put into the form

$$\left.\begin{aligned} \frac{d}{dt}(a\omega_x - k\omega_y - j\omega_z) &= L \\ \frac{d}{dt}(-k\omega_x + b\omega_y - i\omega_z) &= M \\ \frac{d}{dt}(-j\omega_x - i\omega_y + c\omega_z) &= N \end{aligned}\right\}, \qquad (22)$$

which are the same as equations (1), Art. 241. Another form which may be given to the same equations is

$$\left.\begin{aligned} \frac{d}{dt}(A\omega_1 a_1 + B\omega_2 a_2 + C\omega_3 a_3) &= L \\ \frac{d}{dt}(A\omega_1 b_1 + B\omega_2 b_2 + C\omega_3 b_3) &= M \\ \frac{d}{dt}(A\omega_1 c_1 + B\omega_2 c_2 + C\omega_3 c_3) &= N \end{aligned}\right\}, \qquad (23)$$

where a_1, &c., have the same meaning as in Article 253. From these last equations Euler's Equations can be easily deduced.

If O be not fixed in space, equations (22) and (23) still hold good, provided O is the centre of inertia; the *directions* of the axes being fixed in space.

Impulses.—The equations for determining the moments of momentum produced by impulses are

$$(H_1 - H_1') = \underset{\cdot}{L}, \quad (H_2 - H_2') = \underset{\cdot}{M}, \quad (H_3 - H_3') = \underset{\cdot}{N}, \quad (24)$$

where H_1, &c., can be transformed as before.

255. **Impact.**—When two smooth bodies moving in any way collide, the results of the impact are obtained in a manner precisely similar to that employed in Article 226.

In the present case there are thirteen unknown quantities and thirteen equations.

If λ, μ, ν be the angles made by the common normal at the point of contact with axes fixed in space; R the whole impulse of the mutual normal action during the *first period of impact;* p and p' the perpendiculars on its line of action from the centres of inertia of the two bodies; α, β, γ, α', β', γ' the angles made by the axes of the couples produced by R round these points with the principal axes of the bodies: twelve of the equations mentioned above are

$$\left.\begin{array}{ll} \mathfrak{M}\dot{x} = R\cos\lambda, & A\pi_1 = Rp\cos\alpha \\ \mathfrak{M}\dot{y} = R\cos\mu, & B\pi_2 = Rp\cos\beta \\ \mathfrak{M}\dot{z} = R\cos\nu, & C\pi_3 = Rp\cos\gamma \\ \mathfrak{M}'\dot{x}' = -R\cos\lambda, & A'\pi_1' = -Rp'\cos\alpha' \\ \mathfrak{M}'\dot{y}' = -R\cos\mu, & B'\pi_2' = -Rp'\cos\beta' \\ \mathfrak{M}'\dot{z}' = -R\cos\nu, & C'\pi_3' = -Rp'\cos\gamma' \end{array}\right\} \quad (25)$$

where $\dot{x}$, &c., are the changes of the components of the velocity of the centre of inertia of the first body, parallel to axes fixed in space, produced during the first period of impact; π_1, &c., the changes of the angular velocities round the principal axes through the centre of inertia produced during the same period; and $\dot{x}'$, &c., have similar significations for the second body.

At the end of the first period the actual components of the velocity of the centre of inertia of the first body are $\dot{x} + \left(\frac{dx}{dt}\right)_0$, &c., where $\left(\frac{dx}{dt}\right)_0$ represents the component of this velocity immediately before the impact. In like manner, $\pi_1 + \Omega_1$ is the actual angular velocity round the first principal axis. We can then write down, in terms of $\dot{x} + \left(\frac{dx}{dt}\right)_0$, $\pi_1 + \Omega_1$, &c., the relative normal velocity of the points of the two bodies which are in contact. Equating this relative normal velocity to zero gives a thirteenth equation; so that $\dot{\bar{x}}$, π_1, &c., become completely known.

If $\frac{d\bar{x}}{dt}$ be the component of the final velocity assumed by the centre of inertia of the first body at the end of *the second period of impact*, and ω_1 be the final angular velocity round the first principal axis, we have,

$$\left.\begin{aligned} &\frac{d\bar{x}}{dt} - \left(\frac{d\bar{x}}{dt}\right)_0 = (1+e)\,\dot{\bar{x}}, \quad \omega_1 - \Omega_1 = (1+e)\,\pi_1, \text{ \&c.,} \\ &\frac{dx'}{dt} - \left(\frac{d\bar{x}'}{dt}\right)_0 = (1+e)\,\dot{\bar{x}}', \ \ \omega_1' - \Omega_1' = (1+e)\,\pi_1' \text{ \&c.} \end{aligned}\right\}. \quad (26)$$

Since the positions of the two bodies are not sensibly altered during the whole period of impact, it is to be observed that in forming equations (25) and (26) any lines fixed in either body may be taken as fixed in space.

256. **Impulsive Friction.**—When collision takes place between two rough surfaces we can investigate the motion according to the principles laid down in Article 227.

The elementary impulse dF of friction, at each instant of the impact, is to be resolved into two components, dP and dQ, along two tangents through the point of contact at right angles to each other.

If u, v, w be the *relative* tangential and normal velocities at any instant during the impact of the points of the two surfaces which are in contact, u, v, w can be expressed in terms of the velocities of the two centres of inertia and of the angular velocities of the bodies at that instant; they are therefore linear functions of P, Q, R. If slipping take place,

$$\frac{dP}{dQ} = \frac{u}{v}, \quad \text{and} \quad \sqrt{(dP^2 + dQ^2)} = \mu dR.$$

Initially R is zero, and therefore so also are P and Q, except the colliding surfaces be perfectly rough. When $R = R_1$, at the end of the first period of impact, $w = 0$; and if R_2 be the value of R at the end of the whole impact,

$$R_2 = (1+e)\,R_1.$$

If the surfaces which collide be *perfectly rough*, the equations $u = 0$, $v = 0$, $w = 0$ enable us to determine P_1, Q_1, R_1. Knowing the value of R_2 we can find P_2 and Q_2 from the equations $u = 0$, $v = 0$, which hold good throughout the impact.

If the bodies *slip on each other in the same direction* during the whole of the impact, the direction of dF is constant, and we may take $dQ = 0$, $dP = \mu dR$. Hence $P_1 = \mu R_1$, $Q_1 = 0$; which equations, with $w = 0$, determine R_1; then

$$P_2 = \mu\,(1 + e)\,R_1.$$

If a *homogeneous sphere* impinge against a *fixed* surface, or *two homogeneous spheres* collide with each other, we have, by taking as axes of x, y, z parallels to two tangents and the normal at the point of contact,

$$\frac{d\bar{x}}{dt} = \left(\frac{d\bar{x}}{dt}\right)_0 - \frac{P}{m}, \qquad \frac{d\bar{y}}{dt} = \left(\frac{d\bar{y}}{dt}\right)_0 - \frac{Q}{m}, \qquad \frac{d\bar{z}}{dt} = \left(\frac{d\bar{z}}{dt}\right)_0 - \frac{R}{m},$$

$$\frac{d\bar{x}'}{dt} = \left(\frac{d\bar{x}'}{dt}\right)_0 + \frac{P}{m'}, \qquad \frac{d\bar{y}'}{dt} = \left(\frac{d\bar{y}'}{dt}\right)_0 + \frac{Q}{m'}, \qquad \frac{d\bar{z}'}{dt} = \left(\frac{d\bar{z}'}{dt}\right)_0 + \frac{R}{m'},$$

$$\omega_1 = \Omega_1 + \frac{aQ}{I}, \qquad \omega_2 = \Omega_2 - \frac{aP}{I}, \qquad \omega_3 = \Omega_3,$$

$$\omega_1' = \Omega_1' + \frac{a'Q}{I'}, \qquad \omega_2' = \Omega_2' - \frac{a'P}{I'}, \qquad \omega_3' = \Omega_3',$$

where a, m, and I are the radius, mass, and moment of inertia round its diameter, of the first sphere: $\frac{d\bar{x}}{dt}$, &c., the components of the velocity of its centre, ω_1, &c., the components of its angular velocity, at any instant during the impact, $\left(\frac{d\bar{x}}{dt}\right)_0$, &c., and Ω_1, &c., the values of these components immediately before the impact: and a', m', &c., have similar significations for the second sphere.

The velocities of the points of the spheres which are in contact are given by the equations

$$\frac{dx}{dt} = \frac{d\bar{x}}{dt} - \eta\omega_3 + \zeta\omega_2, \qquad \frac{dx'}{dt} = \frac{d\bar{x}'}{dt} - \eta'\omega_3' + \zeta'\omega_2', \text{ \&c.,}$$

where ξ, η, ζ; ξ', η', ζ', are the coordinates of the point of contact relative to the centres of the two spheres; and u, v, w are then determined by the equations

$$u = \frac{dx}{dt} - \frac{dx'}{dt}, \text{ \&c.}$$

Since $\xi = 0,\ \eta = 0,\ \zeta = a,\ \xi' = 0,\ \eta' = 0,\ \zeta' = -a'$, we deduce, from the preceding equations,

$$u = \left(\frac{d\bar{x}}{dt}\right)_0 - \left(\frac{d\bar{x}'}{dt}\right)_0 + a\Omega_2 + a'\Omega_2' - \left(\frac{1}{m} + \frac{1}{m'} + \frac{a^2}{I} + \frac{a'^2}{I'}\right)P,$$

$$v = \left(\frac{d\bar{y}}{dt}\right)_0 - \left(\frac{d\bar{y}'}{dt}\right)_0 - a\Omega_1 - a'\Omega_1' - \left(\frac{1}{m} + \frac{1}{m'} + \frac{a^2}{I} + \frac{a'^2}{I'}\right)Q,$$

$$w = \left(\frac{d\bar{z}}{dt}\right)_0 - \left(\frac{d\bar{z}'}{dt}\right)_0 - \left(\frac{1}{m} + \frac{1}{m'}\right)R,$$

or $$u = u_0 - lP, \quad v = v_0 - lQ, \quad w = w_0 - nR,$$

where l and n are constants.

Hence $du = -l\,dP, \quad dv = -l\,dQ, \quad \dfrac{du}{dv} = \dfrac{dP}{dQ}$;

but, if there be slipping,

$\dfrac{dP}{dQ} = \dfrac{u}{v}$; therefore $\dfrac{du}{dv} = \dfrac{u}{v}$, and accordingly $\dfrac{u}{v}$ is constant, or the direction of slipping is invariable throughout the impact. Moreover, if either u or v vanish so must the other. All slipping then ceases, and cannot recommence, as u and v are independent of R, and friction cannot initiate slipping. Since, in the present case, R_1 is independent of P and Q, if there be no slipping at the end of the impact the result is the same as if there had been no slipping at all.

Hence, in all cases, either the impulse of friction is a maximum, and the direction of slipping the same throughout the impact, or else the surfaces may be regarded as perfectly rough.

If the problem be solved on the latter hypothesis, and the value found for $\sqrt{(P_2^2 + Q_2^2)}$ does not exceed $\mu(1 + e)R_1$, the solution is correct. If $\sqrt{(P_2^2 + Q_2^2)} > \mu(1 + e)R_1$, slipping takes place in the same direction throughout the impact.

EXAMPLES.

Motion of a Body unacted on by Force.

1. The direction of the momentum axis through the centre of inertia is fixed in space, and the moment of momentum is constant.

H_1, H_2, H_3 are constant, and therefore H.

2. The angular velocity at any instant is proportional to the intercept on the instantaneous axis of rotation through the centre of inertia cut off by the momental ellipsoid.

The equation of the momental ellipsoid is $Ax^2 + By^2 + Cz^2 = K$; and r being the intercept on the line whose direction cosines are $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$, we have

$$\frac{K}{r^2} = A\frac{\omega_1^2}{\omega^2} + B\frac{\omega_2^2}{\omega^2} + C\frac{\omega_3^2}{\omega^2} = \frac{T}{\omega^2},$$

where T is the vis viva relative to the centre of inertia; but the velocity of this point being constant as well as the whole vis viva, T is constant, and therefore ω is proportional to r.

3. The component of the angular velocity round the momentum axis through the centre of inertia is constant.

The direction cosines of the momentum axis referred to the principal axes are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$; therefore ϕ being the angle between the momentum axis and the instantaneous axis of rotation,

$$H\omega\cos\phi = A\omega_1^2 + B\alpha_2^2 + C\omega_3^2 = T;$$

but H (Ex. 1) is constant; therefore $\omega\cos\phi$ is constant.

4. If a tangent plane be drawn to the momental ellipsoid at its point of intersection with the instantaneous axis of rotation through the centre of inertia, the distance of this plane from the centre is constant.

$$p = r\cos\phi = \sqrt{\frac{K}{T}}\,\omega\cos\phi = \frac{\sqrt{KT}}{H}.$$

If a body have a fixed point, the results of the preceding examples hold good, the fixed point being substituted for the centre of inertia.

5. A body moves round a fixed point: give a geometrical representation of the motion.

The momental ellipsoid relative to the point rolls on a plane fixed in space, its centre remaining immovable, so that the instantaneous axis of rotation is always the line joining the centre to the point of contact.

6. A body is moving round a fixed point; find the locus of the instantaneous axis of rotation in the body.

Since $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$ are its direction cosines referred to the principal axes through the point, its locus is the cone

$$A(H^2 - AT)x^2 + B(H^2 - BT)y^2 + C(H^2 - CT)z^2 = 0.$$

7. Find the locus of the momentum axis in the body.

As the momentum axis is the perpendicular on the tangent plane at the extremity of the radius vector $\left(\frac{\omega_1}{\omega}, \frac{\omega_2}{\omega}, \frac{\omega_3}{\omega}\right)$ to the momental ellipsoid, its locus is the cone

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{T}{H^2}(x^2 + y^2 + z^2), \quad \text{or}$$

$$\frac{AT - H^2}{A} x^2 + \frac{BT - H^2}{B} y^2 + \frac{CT - H^2}{C} z^2 = 0.$$

8. Determine the curve traced out on the momental ellipsoid by the instantaneous axis.

The equations of the curve are got by combining the equation of the ellipsoid with that of the cone given in Ex. 6; they are, therefore,

$$Ax^2 + By^2 + Cz^2 = K, \qquad A^2x^2 + B^2y^2 + C^2z^2 = \frac{KH^2}{T}.$$

This curve is called the *polhode.*

9. The projections of the polhode on the planes perpendicular to the axes of greatest and least moment of inertia are ellipses. Its projection on the plane perpendicular to the remaining principal axis is a hyperbola.

This appears at once by eliminating x, y, z successively from the two equations of Ex. 8, remembering that $A > B > C$.

10. In what case does the hyperbola become a pair of straight lines?

If $H^2 = BT$.

11. If the body be free, give a geometrical representation of the motion (see Ex. 5).

The momental ellipsoid relative to the centre of inertia rolls on a plane at a constant distance from the centre of inertia and parallel to a plane fixed in space, the instantaneous axis of rotation being the line joining the centre of inertia to the point of contact, whilst the whole system moves with uniform velocity parallel to a fixed direction.

12. Integrate Euler's Equations.

In the present case these equations become

$$A\frac{d\omega_1}{dt} = (B - C)\,\omega_2\,\omega_3, \quad B\frac{d\omega_2}{dt} = (C - A)\,\omega_3\,\omega_1, \quad C\frac{d\omega_3}{dt} = (A - B)\,\omega_1\,\omega_2.$$

Multiply the first by ω_1, the second by ω_2, the third by ω_3, add, and integrate, and we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = T.$$

Again, multiply the first by $A\omega_1$, the second by $B\omega_2$, the third by $C\omega_3$, add, and integrate, and we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2.$$

These equations were obtained before from the equations for the vis viva and the moment of momentum.

Finally, if we multiply the first by $\frac{\omega_1}{A}$, the second by $\frac{\omega_2}{B}$, the third by $\frac{\omega_3}{C}$, and add, we get

$$\omega_1 \frac{d\omega_1}{dt} + \omega_2 \frac{d\omega_2}{dt} + \omega_3 \frac{d\omega_3}{dt} = \left(\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C}\right)\omega_1\,\omega_2\,\omega_3,$$

or

$$\omega \frac{d\omega}{dt} = -\frac{(A-B)(B-C)(C-A)}{ABC}\,\omega_1\,\omega_2\,\omega_3.$$

If we combine the two equations already found with the equation

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2,$$

and solve for ω_1^2, we get

$$\begin{vmatrix} 1, & 1, & 1 \\ A, & B, & C \\ A^2, & B^2, & C^2 \end{vmatrix} \omega_1^2 = \omega^2 BC(C-B) - T(C^2 - B^2) + H^2(C-B);$$

whence

$$\omega_1^2 = \frac{BC}{(A-B)(A-C)}\left\{\omega^2 - \frac{T(B+C)-H^2}{BC}\right\}.$$

If we denote $\frac{T(B+C)-H^2}{BC}$ by λ_1, and the two corresponding quantities by λ_2 and λ_3, we have

$$\omega_1\,\omega_2\,\omega_3 = \frac{ABC}{(A-B)(A-C)(B-C)}\sqrt{\{(\omega^2-\lambda_1)(\lambda_2-\omega^2)(\omega^2-\lambda_3)\}};$$

whence

$$\omega\frac{d\omega}{dt} = \sqrt{\{(\lambda_1-\omega^2)(\lambda_2-\omega^2)(\lambda_3-\omega^2)\}}.$$

13. Show that the three quantities λ_1, λ_2, λ_3 are each positive, and determine the order of their magnitudes.

If we substitute for T and H^2 their values in terms of ω_1, ω_2, ω_3, we get

$$\lambda_1 = \frac{1}{BC}\{A(B+C-A)\,\omega_1^2 + BC\,\omega_2^2 + BC\,\omega_3^2\}.$$

Now, as $\quad B + C - A = 2\int x^2\,dm, \quad B + C > A;$

therefore λ_1 is positive, and similarly for the others. Again,

$$\lambda_1 - \lambda_2 = \frac{A-B}{ABC}(CT - H^2) = \frac{A-B}{ABC}\{A(C-A)\,\omega_1^2 + B(C-B)\omega_2^2\},$$

which is negative, as $A > B > C$.

In like manner $\lambda_3 - \lambda_2$ is negative, thus λ_2 is the greatest, and $\lambda_1 - \lambda_3$ has the same sign as $BT - H^2$, which depends on the initial conditions.

14. Show how to express ω in terms of t by means of an elliptic integral of the first kind.

Since ω_1^2, ω_2^2, ω_3^2 are each positive, $\omega^2 > \lambda_1$, $\omega^2 < \lambda_2$, $\omega^2 > \lambda_3$ (Ex. 12). Hence, if λ be the greater, and λ' the lesser of the two quantities λ_1 and λ_3, w may assume $\omega^2 = \lambda \sin^2\theta + \lambda_2 \cos^2\theta$, and the equation

$$\omega\frac{d\omega}{dt} = \sqrt{\{(\lambda_1-\omega^2)(\lambda_2-\omega^2)(\lambda_3-\omega^2)\}}$$

becomes
$$\frac{d\theta}{dt} = -\sqrt{\{\lambda_2 - \lambda' - (\lambda_2 - \lambda)\sin^2\theta\}}.$$

Hence t is expressed in terms of θ by an integral of the same kind as that which expresses the time in the case of a simple pendulum (Art. 105).

The student will observe that ω_1, ω_2, ω_3, are expressed in terms of θ by the equations

$$\omega_1 = \sqrt{\frac{H^2 - CT}{A(A-C)}}\cos\theta, \quad \omega_2 = \sqrt{\frac{H^2 - CT}{B(B-C)}}\sin\theta, \quad \omega_3 = \sqrt{\frac{AT - H^2}{C(A-C)}}\,\Delta(k, \theta),$$

where
$$k^2 = \frac{(A-B)(H^2 - CT)}{(B-C)(AT - H^2)}, \quad \text{if } \lambda_1 > \lambda_3;$$

and by the equations

$$\omega_1 = \sqrt{\frac{H^2 - CT}{A(A-C)}}\,\Delta(k,\theta), \quad \omega_2 = \sqrt{\frac{AT - H^2}{B(A-B)}}\sin\theta, \quad \omega_3 = \sqrt{\frac{AT - H^2}{C(A-C)}}\cos\theta,$$

where
$$k^2 = \frac{(B-C)(AT - H^2)}{(A-B)(H^2 - CT)}, \quad \text{if } \lambda_3 > \lambda_1.$$

15. If a body be moving in any manner, the momentum axis, and the instantaneous axis of rotation through a given point of the body, are the radius vector and the perpendicular on the corresponding tangent plane of the ellipsoid of gyration (see *Integral Calculus*, Art. 216) relative to the point.

This is the reciprocal of the theorem given in Art. 244. It can be easily proved directly: σ_1, σ_2, σ_3 being the angles made by the momentum axis with the principal axes, and a, b, c the principal radii of gyration,

$$H\cos\sigma_1 = A\omega_1 = \mathfrak{M}a^2\omega\cos\alpha, \text{ \&c};$$

but, if x', y', z' be a point on the ellipsoid of gyration, α, β, γ the angles made with the axes by the perpendicular on the corresponding tangent plane, and p the length of the perpendicular, $x'p = a^2\cos\alpha$; therefore, &c.

The student will observe that the results obtained in this Example, and in Examples 16, 17, and 18, hold good for a body moving *under the action of any forces.*

16. If a tangent plane to the ellipsoid of gyration relative to any point of a body be drawn at right angles to the instantaneous axis of rotation passing through the point, and ϕ be the angle between the instantaneous axis and the radius vector to the point of contact, $\omega = \dfrac{H\cos\phi}{\mathfrak{M}p^2}$.

From the equations of the last example, if R be the radius vector of the ellipsoid of gyration,

$$pR = \frac{H}{\mathfrak{M}\omega}; \quad \text{but } p = R\cos\phi; \quad \text{therefore, \&c.}$$

17. If a body be rotating round a fixed point, or a free body round its centre of inertia, the couple resulting from centrifugal forces lies in the plane containing the momentum axis and the instantaneous axis of rotation, and its magnitude is $\mathfrak{M}\omega^2 Rp\sin\phi$, where R, p, and ϕ have the same significations as in the last Example.

The components of the couple resulting from centrifugal forces are (Art. 242)

$$(B - C)\,\omega_2\,\omega_3, \quad (C - A)\,\omega_3\,\omega_1, \quad (A - B)\,\omega_1\,\omega_2,$$

or $\mathfrak{M}\,\omega^2\,(b^2 - c^2)\cos\beta\cos\gamma$, $\mathfrak{M}\,\omega^2\,(c^2 - a^2)\cos\gamma\cos\alpha$, $\mathfrak{M}\,\omega^2\,(a^2 - b^2)\cos\alpha\cos\beta$;

but double the projection of the triangle formed by the origin, x', y', z', and $p\cos\alpha$, $p\cos\beta$, $p\cos\gamma$, is

$$p\,(x'\cos\beta - y'\cos\alpha), \quad \text{or} \quad (a^2 - b^2)\cos\alpha\cos\beta,$$

and double the area of the same triangle is $Rp\sin\phi$; therefore, &c.

The components of the centrifugal couple may be obtained directly in a manner somewhat different from that given, Art. 242.

If x, y, z be the coordinates of any point of the body, and q the perpendicular from it on the instantaneous axis, $\omega^2\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\,q\,dm$ is the moment of the centrifugal force exerted by dm. The axis of the corresponding couple is at right angles to the lines (x, y, z) and (α, β, γ), and its direction angles λ, μ, ν are therefore given by the equations

$$\frac{y\cos\gamma - z\cos\beta}{\cos\lambda} = \frac{z\cos\alpha - x\cos\gamma}{\cos\mu} = \frac{x\cos\beta - y\cos\alpha}{\cos\nu}.$$

Since $q^2 = x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2$, the common value of the above fractions is q; and the components of the centrifugal couple resulting from dm are

$$\omega^2\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\,(y\cos\gamma - z\cos\beta)\,dm,$$

$$\omega^2\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\,(z\cos\alpha - x\cos\gamma)\,dm,$$

$$\omega^2\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\,(x\cos\beta - y\cos\alpha)\,dm.$$

If we integrate these expressions, and remember that

$$\int xy\,dm = \int yz\,dm = \int zx\,dm = 0,$$

we get the values already obtained.

18. If a tangent plane be drawn to the ellipsoid of gyration at the point where it is met by the axis of the centrifugal couple, the perpendicular on this tangent plane is the axis of the rotation produced by the centrifugal couple.

L', M', N' being the components of the centrifugal couple, and $\delta\omega_1$, $\delta\omega_2$, $\delta\omega_3$, the rotations produced by it *considered alone*, we have, from Euler's Equations,

$$A\delta\omega_1 = L'dt, \quad B\delta\omega_2 = M'dt, \quad C\delta\omega_3 = N'dt;$$

but these equations are of the same form as those connecting the instantaneous axis with the components of the couple of principal moments; therefore, &c.

It follows from this, that the axis of rotation produced by the centrifugal couple is at right angles to the momentum axis: for, if OR be the momentum axis; OP the instantaneous axis of rotation; OR' the axis of the centrifugal couple, and OP' the axis of the centrifugal couple rotation; OR', being at right angles to OP (Ex. 17), is conjugate to OR: hence OR is parallel to the tangent plane through R', and therefore at right angles to OP'. Also, OR and OR' are the principal axes of the section of the ellipsoid made by their plane.

19. If a body be rotating round a fixed point, or a free body round its centre of inertia, under the action of no forces, the component of the angular velocity round the momentum axis is constant, and the intercept on the momentum axis by the ellipsoid of gyration is of constant length R.

These results follow at once from Ex. 18 and 16.

20. The momentum axis describes in the body the cone which passes through the intersection of the ellipsoid of gyration and the sphere whose radius is R.

21. To determine the nature of the projections on a cyclic plane, by lines parallel to the greatest and least axes of the ellipsoid of gyration, of the spherical conic described by the momentum axis.

The projections of the spherical conic on a cyclic plane may readily be obtained from its orthogonal projections on the planes of yz and xy. The projection on the plane of yz is found by eliminating x between the equations of the ellipsoid and the sphere $x^2 + y^2 + z^2 = R^2$: it is

$$\frac{y^2(a^2 - b^2)}{(a^2 - R^2)b^2} + \frac{z^2(a^2 - c^2)}{(a^2 - R^2)c^2} = 1.$$

If θ be the angle which a cyclic plane makes with the plane of yz, it is easily seen that

$$\cos\theta = \frac{c}{b}\sqrt{\frac{a^2 - b^2}{a^2 - c^2}};$$

hence lines parallel to x will project the ellipse, whose equation has just been given, into a circle; since lines parallel to the axis of y remain unaltered, and those parallel to the axis of z are increased in the ratio $1 : \cos\theta$. If we call ρ the radius of this circle,

$$\rho^2 = b^2\,\frac{a^2 - R^2}{a^2 - b^2}.$$

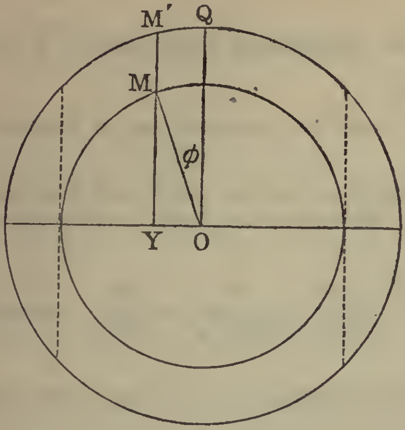
In like manner, the projection on the same cyclic plane by lines parallel to z is a circle whose radius is ρ', where

$$\rho'^2 = b^2\,\frac{R^2 - c^2}{b^2 - c^2}.$$

If $R > b$, the spherical conic traced by the momentum axis *drawn in one direction* will be round the greatest axis of the ellipsoid, and lines parallel to z each meet it in two points, while those parallel to x meet it in one point. Moreover, the two circles formed by its projections on the cyclic plane are concentric, the larger one being formed by lines parallel to z. Also the plane containing the two projecting lines drawn from any point on the spherical conic meets the cyclic plane in a line at right angles to the axis of y, so that the two projections of a point on the spherical conic lie on the same ordinate of the circles, at right angles to the axis of y; and the projection on the inner circle performs a complete revolution, as the momentum axis moves round the spherical conic, while that on the outer circle oscillates between the tangents to the inner circle drawn at the extremities of the y diameter. If $R < b$, z takes the place of x, and x of z.

22. To determine the position of the momentum axis in the body at any time.

This may be done by determining the position of the projection of the



extremity of the momentum axis. Let us suppose that $R > b$, so that the inner circle is the projection by lines parallel to x. Let M and M' be the positions of the projections of the extremity of the momentum axis at any time. Let $MY = \xi$, $M'Y = \zeta$, $OY = y$. If ds be the circular arc described by M in the time dt,

$$\frac{ds}{dy} = \frac{\rho}{\xi}, \quad \text{whence} \quad \rho\frac{d\phi}{dt} = \frac{\rho}{\xi}\frac{dy}{dt}.$$

Now $$\frac{dy}{dt} = -z\omega_1 + x\omega_3 = \omega(x\cos\gamma - z\cos\alpha) = \omega pxz\left(\frac{1}{c^2} - \frac{1}{a^2}\right) \quad \text{(Ex. 15)};$$

and θ being the same as in the last example, $x = \zeta\sin\theta$, $z = \xi\cos\theta$, also (Ex. 16)

$$p\omega = \frac{H}{\mathfrak{M}R};$$

whence, finally,

$$\frac{d\phi}{dt} = \zeta\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\frac{H}{\mathfrak{M}R}\,\frac{ac}{b^2}\,\frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{a^2-c^2}$$

$$= \frac{H}{\mathfrak{M}R}\,\frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{ab^2c}\,\zeta = \frac{H}{\mathfrak{M}R}\,\frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{ab^2c}\sqrt{\rho'^2 - \rho^2\sin^2\phi}$$

$$= \frac{H}{\mathfrak{M}R}\,\frac{\sqrt{(a^2-b^2)(b^2-c^2)}}{ab^2c}\,\rho'\sqrt{1 - \frac{\rho^2}{\rho'^2}\sin^2\phi}.$$

In like manner, if $QOM' = \psi$, we have

$$\frac{d\psi}{dt} = \frac{H}{\mathfrak{M}R}\,\frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{ab^2\,c}\,\rho\sqrt{1 - \frac{\rho'^2}{\rho^2}\sin^2\psi}.$$

If $\rho' > \rho$, as in the case supposed, the expression for $\frac{d\phi}{dt}$ gives t as an elliptic integral of the first kind.

If $R < b$, then $\rho > \rho'$ and t is expressed as an elliptic integral by means of ψ.

23. A body is rotating round a fixed point, or a free body round its centre of inertia: if $H^2 = BT$, determine the position of the momentum axis in the body at any time.

By Ex. 16, $$R = \frac{H}{\mathfrak{M}\omega p} = \frac{H}{\sqrt{\mathfrak{M}^2 p^2 \omega^2}} = \frac{H}{\sqrt{\mathfrak{M}T}};\ \text{whence}\quad R = b.$$

In this case the momentum axis lies in one of the cyclic planes of the ellipsoid of gyration.

Again,

$$\frac{d\phi}{dt} = \frac{H}{\mathfrak{M}}\,\frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{ab^2\,c}\cos\phi;\ \text{and putting}\ \frac{H}{\mathfrak{M}}\,\frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{ab^2\,c} = k,$$

we obtain $$\cot\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = \cot\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)e^{kt}.$$

Hence the momentum axis, and therefore the instantaneous axis, tends to approach without limit to the mean radius of gyration.

24. If the moment of momentum divided by the square root of the product of the mass and vis viva be nearly equal to the greatest or least radius of gyration; determine the motion.

In the first case R nearly $= a$; then the circle produced by projecting by lines parallel to x is in general small, and the circle produced by lines parallel to z is in general large, and the projection of the extremity of the momentum axis vibrates only through a small arc of the latter.

Hence, in the equations of Ex. 22, we may take $\sin\psi = \psi$. We then obtain, k having the same signification as in the last example,

$$\frac{\rho'}{\rho}\psi = \sin\left(\frac{k\rho'}{R}t + \epsilon\right).$$

If t_1 denote the time which the body takes to revolve round its axis of rotation (which is approximately constant and coincident with the greatest axis of the ellipsoid of gyration), and t_2 the time of a complete revolution or oscillation of the momentum axis round the axis of x; since approximately

$$\frac{H}{\mathfrak{M}a^2}\,t_1 = 2\pi, \qquad \frac{k\rho'_1}{R}\,t_2 = 2\pi, \text{ and } R = a,$$

we have
$$t_2 = t_1\,\frac{bc}{\sqrt{(a^2-b^2)(a^2-c^2)}}.$$

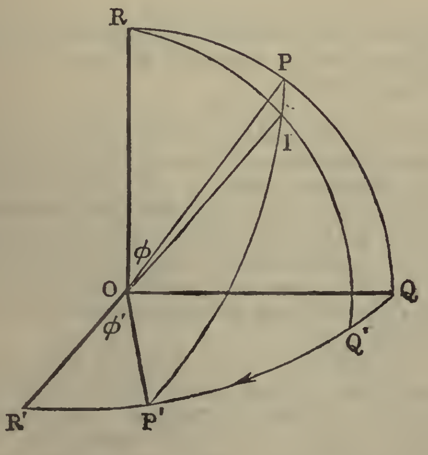
A corresponding result and equation may be obtained in the second case when R nearly $= c$.

The first case would be applicable to the Earth, if its axis of rotation were nearly but not exactly a principal axis. In this case t_1 would be the length of the day. Of course the attractions of other bodies are not taken into account.

25. The motion of the momentum axis in the body consists of a series of rotations, the axis of each rotation being at right angles to the momentum axis and the centrifugal couple axis, and the magnitude of the rotation being equal and opposite to the rotation of the body round the same axis.

The existence of the centrifugal couple tends at each instant to alter the position of the momentum axis, since the new moment of momentum is the resultant of the principal couple at the beginning of the instant and the momentum produced by the centrifugal couple during the instant. The former component is H, the latter $H\omega \sin\phi\, dt$ (Ex. 17), and the two are at right angles. Hence the momentum axis turns towards the centrifugal couple axis with an angular velocity $\omega \sin\phi$, which is equal and opposite to the angular velocity of the body round the perpendicular to the momentum axis and the centrifugal couple axis.

26. The axis of the centrifugal couple, regarded as a radius vector of the ellipsoid of gyration, describes areas proportional to the time in the invariable plane of principal moments.



Let OR and OR' be the momentum axis and the centrifugal couple axis.

Describe a sphere round O as centre, and let OP and OP' be the positions of the actual axis of rotation, and of the axis of rotation due to centrifugal forces corresponding to OR and OR'. Let the lengths of OR and OR', regarded as radii vectores of the ellipsoid of gyration, be R and r. As we have seen (Ex. 18), OR' is at right angles to OR and OP, and OR is at right angles to OP'. Let the body be rotating clockwise round OP at the beginning of dt; at its end, OR in the body will have turned towards OR'; but the rotation of the body round OQ being equal and opposite, OR remains fixed in space. Again, OP in the body will have turned towards OP', into the position OI, and as the body is rotating round OP, its motion cannot alter the position of OI in space by more than an infinitely small quantity of the second order.

Hence, in the figure the line OR and the plane at right angles to it are fixed in space, whilst OP and OI are the positions in space of the instantaneous axis of rotation at the beginning and end of the time dt. The position of OR' at the end of the time dt is at right angles to the plane RIQ'. If dv be the angle described by OR' in the time dt, $dv = QOQ'$. If ω' be the angular velocity produced in the time dt by the centrifugal couple, we have, by Ex. 16, 17, and 18,

$$\mathfrak{M}\omega' p' r = \mathfrak{M}\omega^2 pR \sin\phi \, dt;$$

whence
$$\frac{\omega R p \sin\phi \, dt}{rp'} = \frac{\omega'}{\omega} = \frac{\sin PI}{\sin P'I} = \frac{\sin PI}{\sin IR} \cdot \frac{\sin IR}{\sin P'I}$$

$$= \frac{dv}{\sin P'PQ} \cdot \frac{\sin RP}{\sin P'P} = \frac{dv \sin\phi}{\cos\phi'};$$

but
$$p' = r\cos\phi', \quad \text{and} \quad p = R\cos\phi,$$

whence, finally,
$$r^2 \, dv = \omega R^2 \cos\phi \, dt;$$

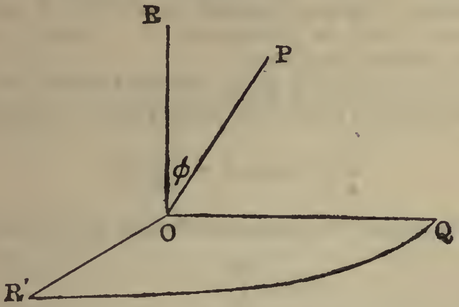
and as R and $\omega\cos\phi$ are each constant, the theorem is proved. It is to be observed that it follows from the equations of Ex. 15 and 18 that the angles ROP and $R'OP'$ are each acute.

27. To determine the position of the body in space at any time.

The position of the momentum axis in the body, or in other words the line in the body which at a given time coincides with the momentum axis, is known. If, then, we make this line of the body coincide with the momentum axis (whose position in space remains unaltered), and then turn the body through the proper angle round this axis, the position of the body in space is determined.

In order to effect the latter part of the determination, we must consider the position in space of the line which is at right angles to the momentum axis and the axis of the centrifugal couple.

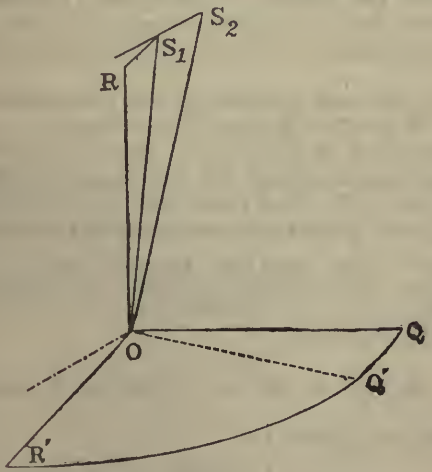
The momentum axis describes a cone C in the body, and the lines at right angles to the tangent planes to this cone C form a reciprocal cone C', one of whose edges always lies in the plane of principal moments, and is at right angles to the momentum axis and the centrifugal couple axis. The position of the momentum axis in the body being known, so likewise is the position of the corresponding edge of C', and if we can determine the position of the latter in space, the problem is solved. Now, OR being the momentum axis, and OQ the corresponding edge of C', the instantaneous axis of rotation OP lies in their plane; and the angular velocity ω round OP is equivalent to $\omega \cos\phi$ round OR, and $\omega \sin\phi$ round OQ. If, then, we suppose the cone C' rigidly connected with



the body, the whole motion consists of the rolling and sliding of the cone C' on the plane of principal moments. The angular velocity of sliding is $\omega \cos\phi$, and if $d\epsilon$ be the angle between two consecutive edges of the cone C', the velocity with which OQ turns in consequence of the rolling is $\frac{d\epsilon}{dt}$. Hence the whole angular velocity of OQ round OR is

$$\omega \cos\phi - \frac{d\epsilon}{dt},$$

if the cone C' be on the same side of the plane of principal moments as OP. Thus, if the rotation round OP be counter-clockwise, $\omega \cos\phi$ will be from R' to Q, whilst the rolling round OQ will bring an edge into contact with the plane of principal moments which is nearer to OR', and this will impart an angular velocity $-\frac{d\epsilon}{dt}$ to the edge of C', which is in the plane of principal moments. We



may arrive at the same result in another way, by considering the motion of the cone C regarded as rigidly attached to the body. The rotation round OQ

(supposed counter-clockwise) brings the line OS_1 consecutive to OR into the position OR, and moves the next consecutive tangent plane of C in a direction parallel to OR'. The next rotation is effected round a line at right angles to $OS_1 S_2$, which is therefore nearer than OQ to OR': thus the motion of OQ, in consequence of a counter-clockwise rotation round OQ, is clockwise. On the other hand, the motion of OQ, in consequence of a counter-clockwise rotation round OR, is likewise counter-clockwise.

Hence, on the whole, if υ be the angle described by the line at right angles to the momentum axis and the centrifugal couple axis in the fixed plane,

$$\upsilon = \int \omega \cos \phi \, dt - \int \frac{d\epsilon}{dt} dt.$$

Since $d\epsilon$ is the angle between two consecutive edges of the cone C',

$$d\epsilon = \frac{ds}{R},$$

if s be the arc of the spherical conic in which the cone C' meets the sphere of radius R; whence, finally,

$$\upsilon = \omega t \cos \phi - \frac{s}{R}.$$

If the cone C' be on the opposite side of the plane of principal moments from OP and OR, or, in other words, if the curvature of the cone C turned towards OP be convex instead of concave, the two parts of the motion of OQ have the same sign, and

$$\upsilon = \omega t \cos \phi + \frac{s}{R}.$$

The former case occurs when R is less than the mean axis of the ellipsoid of gyration; the latter, when it is greater. In either case s is determined by knowing the position of OQ in the cone C'.

In order to facilitate the drawing of the figures, the rotation is in Ex. 26 supposed to be clockwise, but in the present example counter-clockwise.

Examples 15 to 27 contain the substance of Professor Mac Cullagh's Lectures on Rotation.

28. The normals to the cone described by the instantaneous axis of rotation intersect the ellipsoid of gyration in a line of curvature.

The normal to the plane of OP and OP' (figure of Ex. 26) is conjugate to OR and OR', and hence passes through the intersection of the ellipsoid of gyration with the confocal $a^2 - R^2$. This result follows also from Ex. 6.

29. Show that, if υ and ϕ have the same meaning as in Ex. 26,

$$\frac{d\upsilon}{dt} = \frac{T}{H} + \frac{(AT - H^2)(BT - H^2)(CT - H^2)}{ABCHT^2} \cot^2 \phi.$$

Substituting, by Ex. 3 and 16,

$$\omega \cos \phi \text{ for } \frac{T}{H}, \; \mathfrak{M}a^2 \text{ for } A, \; \mathfrak{M}b^2 \text{ for } B, \; \mathfrak{M}c^2 \text{ for } C, \text{ and } R^2 \text{ for } \frac{H^2}{\mathfrak{M}T},$$

the equation given above is reduced to

$$\frac{d\upsilon}{dt} = \omega \cos \phi \left\{ 1 + \frac{(a^2 - R^2)(b^2 - R^2)(c^2 - R^2)}{a^2 b^2 c^2} \cot^2 \phi \right\}.$$

Now, by Ex. 26, $$\frac{dv}{dt} = \omega \cos \phi \frac{R^2}{r^2},$$

and $$\frac{R^2}{r^2} = 1 + \frac{(a^2 - R^2)(b^2 - R^2)(c^2 - R^2)}{a^2 b^2 c^2} \cot^2 \phi,$$

as may be thus proved.

The axes of a central section of the quadric a, b, c are parallel to the normals to the two confocals through the extremities of the semi-diameter D conjugate to the section, and (a', a'' being the semi-axes of the confocals) are given by the equations

$$R^2 = a^2 - a'^2, \quad r^2 = a^2 - a''^2$$

(Salmon's *Geometry of Three Dimensions*, § 164). Moreover, if r' be the axis normal to a'' of the section conjugate to D in the quadric a', the direction of r' coincides with that of r, and the magnitude of r' is given by the equation

$$r'^2 = a'^2 - a''^2 = r^2 - R^2.$$

Again, if p' be the perpendicular on the tangent plane to a', which is parallel to the plane of D and r' or r,

$$p'^2 = a'^2 \cos^2 \alpha + b'^2 \cos^2 \beta + c'^2 \cos^2 \gamma = p^2 - R^2,$$

since $$a'^2 = a^2 - R^2, \quad b'^2 = b^2 - R^2, \quad c'^2 = c^2 - R^2.$$

Hence, if ψ be the angle between D and the direction of r or r', we have, from the quadric a,

$$D^2 = \frac{a^2 b^2 c^2}{r^2 p^2 \sin^2 \psi}$$

and, from the quadric a',

$$D^2 = \frac{a'^2 b'^2 c'^2}{r'^2 p'^2 \sin^2 \psi};$$

and therefore

$$\frac{a^2 b^2 c^2}{r^2 p^2} = \frac{(a^2 - R^2)(b^2 - R^2)(c^2 - R^2)}{(r^2 - R^2)(p^2 - R^2)}; \quad \text{but} \quad \frac{p^2}{R^2 - p^2} = \cot^2 \phi;$$

whence, finally, $$\frac{R^2}{r^2} - 1 = \frac{(a^2 - R^2)(b^2 - R^2)(c^2 - R^2)}{a^2 b^2 c^2} \cot^2 \varphi.$$

The expression given in this Example for $\frac{dv}{dt}$ is due to Poinsot.

30. Determine the differential equation of the curve traced out on the invariable plane by the point of contact of the momental ellipsoid (*see* Ex. 5).

If ρ and v be the polar coordinates of the point of contact, the origin being the foot of the perpendicular from the fixed point on the invariable plane,

$$\rho^2 = r^2 - p^2,$$

where $$r = \sqrt{\frac{K}{T}}\,\omega, \quad p = \frac{\sqrt{KT}}{H} \text{ (Ex. 2, 4), and } \frac{dv}{dt} = \frac{T}{H}\left(1 + \mu \frac{p^2}{\rho^2}\right)$$

where $$\mu = \frac{(AT - H^2)(BT - H^2)(CT - H^2)}{ABCT^3} \text{ (Ex. 29).}$$

If we express ω in terms of ρ we get, on substituting in the final equation of Ex. 12,

$$\frac{d\rho}{dv} = \frac{KH}{T^2} \cdot \frac{\rho}{\rho^2 + \mu p^2} \sqrt{\left\{ \left[\lambda_1 - \frac{T}{K}(p^2 + \rho^2) \right] \left[\lambda_2 - \frac{T}{K}(p^2 + \rho^2) \right] \left[\lambda_3 - \frac{T}{K}(p^2 + \rho^2) \right] \right\}}.$$

The curve whose differential equation is here investigated is called the *herpolhode.*

31. Show that the herpolhode lies between two circles, the squares of whose radii are

$$\frac{K}{T}\left(\lambda_2 - \frac{T^2}{H^2}\right), \text{ and } \frac{K}{T}\left(\lambda_1 - \frac{T^2}{H^2}\right) \text{ or } \frac{K}{T}\left(\lambda_3 - \frac{T^2}{H^2}\right),$$

according as λ_1 is greater or less than λ_3.

By Ex. 30

$$\rho^2 = \frac{K}{T}\left(\omega^2 - \frac{T^2}{H^2}\right).$$

Now (Ex. 14) $\qquad \omega^2 > \lambda_1, \quad \omega^2 < \lambda_2, \quad \omega^2 > \lambda_3.$

Hence the greatest and least values of ρ^2 are comprised between the limits given above.

32. If two of the principal moments of inertia of the body be equal, prove that—(1) The simultaneous positions of the momentum axis and the instantaneous axis of rotation lie in a plane containing the axis of unequal moment of inertia; (2) The instantaneous axis and the momentum axis describe in the body right circular cones whose semi-angles are i and γ, where

$$\tan^2 i = \frac{C}{A} \cdot \frac{H^2 - CT}{AT - H^2}, \text{ and } \tan^2 \gamma = \frac{A}{C} \cdot \frac{H^2 - CT}{AT - H^2},$$

the axis of unequal moment of inertia being the axis of z; (3) The values of ω, ω_1, ω_2, ω_3 at any time are given by the equations

$$\omega_1 = \omega \sin i \cos \left(\frac{A - C}{A} \omega_3 t + \chi\right), \quad \omega_2 = -\omega \sin i \sin \left(\frac{A - C}{A} \omega_3 t + \chi\right),$$

$$\omega_3 = \omega \cos i, \quad \omega^2 = \frac{(A + C) T - H^2}{AC}, \quad \text{where } \chi \text{ is an arbitrary constant.}$$

Motion of a Body under the Action of Force.

33. A body is given a rotation round a principal axis through its centre of inertia, and is acted on by a couple having this line for its axis. Show that the body will continue to revolve round the axis of initial rotation.

34. A free body is set in motion by an impulse. If the initial motion be a pure rotation, show that the directions of the impulse and instantaneous axis of rotation are principal axes of a section of the momental ellipsoid relative to the centre of inertia.

Since the initial motion is a pure rotation, the initial velocity of the centre of inertia, and therefore the direction of the impulse, is at right angles to the direction of the instantaneous axis of rotation. The statement above then follows from Art. 244.

35. On the same hypothesis as in the last example, show that the instantaneous axis of rotation is a principal axis of the body, at the point in which it is met by its shortest distance from the line of direction of the impulse.

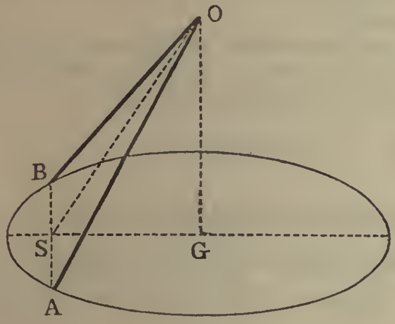
36. If different impulses produce velocities of rotation round parallel instantaneous axes, prove that in general these axes lie in one plane containing the centre of inertia, and perpendicular to the lines of direction of the impulses, and that the locus of the points in which this plane meets these lines is a straight line.

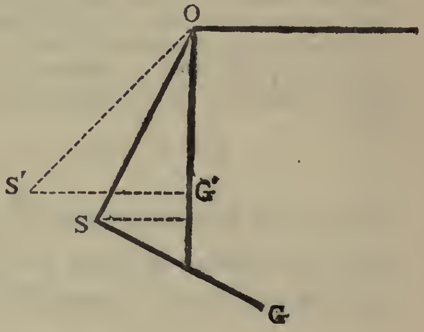
37. If in the preceding example the instantaneous axes are parallel to a principal axis through the centre of inertia, prove that the lines of direction of the impulses lie in the corresponding principal plane at the centre of inertia.

The theory of the centre of percussion, given in Art. 218, may be collected from Examples 35, 36, 37.

38. A circular disk is hung, with its plane horizontal, from a fixed point vertically over its centre by means of three equal strings attached to three fixed points in the circumference of the disk at equal distances from each other. One of the strings is cut: determine the initial tensions of the other two.

The two tensions along the threads OA and OB may be replaced by the single force F along OS, where $F = 2T \cos AOS$, S being the middle point of the chord joining the fixed points A and B.





Let the length $OS = l$, $SG = b$, where G is the centre of the disk. Let OS and SG at any time make the angles α and ψ with the horizontal line which is the initial direction of SG; then ψ is obviously the angle through which the disk has turned. Also, if x, y, be the coordinates of G referred to a horizontal and vertical line through O,

$$x = b\cos\psi - l\cos\alpha, \quad y = b\sin\psi + l\sin\alpha.$$

The dynamical equations are

$$M\frac{d^2y}{dt^2} = Mg - F\sin\alpha, \quad M\frac{d^2x}{dt^2} = F\cos\alpha, \quad Mk^2\frac{d^2\psi}{dt^2} = Fb\sin(\psi + \alpha),$$

as the whole angular velocity is round an axis parallel to AB.

If we differentiate the expressions for x and y twice, we obtain expressions for

$$\frac{dx}{dt}, \ \frac{d^2x}{dt^2}, \ \frac{dy}{dt}, \ \frac{d^2y}{dt^2}.$$

To determine the initial value of F, since the body starts from rest, put

$$\frac{dx}{dt}=0,\quad \frac{dy}{dt}=0,\quad \psi=0 \text{ in these four equations ; we get thus}$$

$$\frac{d\alpha}{dt}=0,\quad \frac{d\psi}{dt}=0,\quad \frac{d^2x}{dt^2}=l\sin\alpha\frac{d^2\alpha}{dt^2},\quad \frac{d^2y}{dt^2}=l\cos\alpha\frac{d^2\alpha}{dt^2}+b\frac{d^2\psi}{dt^2};$$

substituting these values in the dynamical equations and solving for F, we obtain, since initially $b = l\cos\alpha$,

$$F = Mg\frac{k^2\sin\alpha}{k^2+l^2\sin^2\alpha\cos^2\alpha}.$$

This example is taken from Routh's *Rigid Dynamics.*

39. One end of a heavy rod rests on a horizontal plane and against a vertical wall; the other rests against a parallel vertical wall. All the surfaces being smooth, if the rod slips down, determine the motion.

Take the intersection of the horizontal and vertical planes passing through the first end of the rod for axis of x, and a vertical plane, at right angles to the walls and passing through the initial position of the centre of inertia of the rod, for the plane of yz, the axis of z being vertical.

Let β be the angle which the rod at any time makes with the axis of y, $2a$ its length, $2b$ the distance between the walls, x_1, y_1, z_1; x_2, y_2, z_2; and $\bar{x}, \bar{y}, \bar{z}$ the coordinates of the two extremities and of the centre of inertia of the rod. Then

$$y_1=0,\quad z_1=0,\quad y_2=2b,\quad \bar{y}=\tfrac{1}{2}(y_1+y_2)=b:$$

also $\bar{y}=a\cos\beta$, whence $\cos\beta=\frac{b}{a}$; thus, as β is constant, the motion of the rod relative to its centre of inertia is a rotation round the axis of y, whose magnitude at any time may be denoted by ϕ. Again, as $m\frac{d^2\bar{x}}{dt^2}=0$, and as the initial value of $\frac{d\bar{x}}{dt}$ is zero, $\frac{d\bar{x}}{dt}=0$; also, since $\bar{y}$ is constant, $\frac{d\bar{y}}{dt}=0$; whence the equation of vis viva is

$$m\left\{k^2\left(\frac{d\phi}{dt}\right)^2+\left(\frac{d\bar{z}}{dt}\right)^2\right\}=2mg(\bar{z}_0-\bar{z}).$$

Now $$k^2=\frac{a^2\sin^2\beta}{3},\quad \bar{z}=a\sin\beta\cos\phi,$$

whence, as the initial value of ϕ is zero, we obtain

$$(1+3\sin^2\phi)\left(\frac{d\phi}{dt}\right)^2=\frac{6g}{\sqrt{a^2-b^2}}(1-\cos\phi).$$

40. A homogeneous sphere, having an initial angular velocity round a horizontal axis, is projected along a rough horizontal plane: determine the motion, neglecting the couple of rolling friction. (Jellett, *Theory of Friction*, Chap. v.).

The axes being three mutually perpendicular straight lines through the centre whose directions are fixed in space, equations (22) of Art. 254, become for a homogeneous sphere whose radius is r,

$$\tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_1}{dt} = L, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_2}{dt} = M, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_3}{dt} = N.$$

Let X and Y be the components of friction along two horizontal axes, x and y the coordinates of the centre of the sphere, u and v the components of the velocity of its point of contact with the rough plane; then, by Art. 235,

$$u = \frac{dx}{dt} - r\omega_2, \quad v = \frac{dy}{dt} + r\omega_1.$$

The equations of motion are

$$\mathfrak{M}\frac{d^2x}{dt^2} = X, \quad \mathfrak{M}\frac{d^2y}{dt^2} = Y, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_1}{dt} = rY, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_2}{dt} = -rX.$$

Combining the first and last of these, we have

$$\frac{du}{dt} = \frac{d^2x}{dt^2} - r\frac{d\omega_2}{dt} = \tfrac{7}{2}\frac{X}{\mathfrak{M}}.$$

In like manner,

$$\frac{dv}{dt} = \tfrac{7}{2}\frac{Y}{\mathfrak{M}}, \text{ whence } \frac{du}{dv} = \frac{X}{Y}.$$

Now if there be slipping,

$$X = -\mu\mathfrak{M}g\frac{u}{\sqrt{(u^2+v^2)}}, \quad Y = -\mu\mathfrak{M}g\frac{v}{\sqrt{(u^2+v^2)}}, \text{ and } \frac{X}{Y} = \frac{u}{v};$$

whence $\dfrac{u}{v} = \text{constant} = \cot\alpha = \dfrac{a - r\Omega_2}{b + r\Omega_1}$, where a, b, Ω_1, Ω_2 are the initial values of $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, ω_1, and ω_2. Hence

$$X = -\mu\mathfrak{M}g\cos\alpha, \quad Y = -\mu\mathfrak{M}g\sin\alpha.$$

These are the components of a constant force having a fixed direction, which is inclined to the direction of the initial velocity of the centre, unless $a\Omega_1 + b\Omega_2 = 0$. If this equation holds good, the centre of the sphere moves along a straight line, to which the initial axis of rotation is at right angles. If this condition be not fulfilled, the centre of the sphere describes a parabola.

Substituting the values of X and Y in the equations of motion, we find that slipping ceases along the axis of x when $t = \dfrac{2(a - r\Omega_2)}{7\mu g\cos\alpha}$, and along the axis of y when $t = \dfrac{2(b + r\Omega_1)}{7\mu g\sin\alpha}$; but $\dfrac{b + r\Omega_1}{\sin\alpha} = \dfrac{a - r\Omega_2}{\cos\alpha}$, hence, slipping along each axis ceases at the same time, t_0, where

$$t_0 = \tfrac{2}{7}\frac{\sqrt{\{(a - r\Omega_2)^2 + (b + r\Omega_1)^2\}}}{\mu g}.$$

After pure rolling begins it will continue, since the values which X and Y must take in order to maintain it are zero; the components of the velocity of the centre are then given by the equations

$$\frac{dx}{dt} = \frac{5a + 2r\Omega_2}{7}, \quad \frac{dy}{dt} = \frac{5b - 2r\Omega_1}{7}.$$

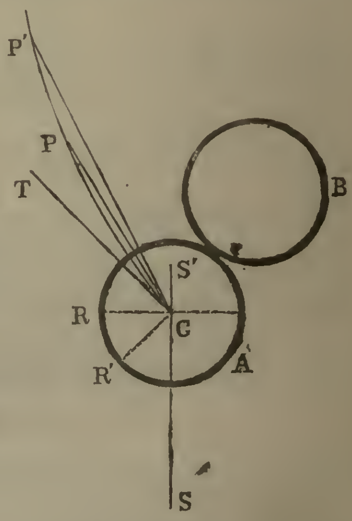
If β be the angle which the final line of motion of the ball makes with the axis of x,

$$\tan\beta = \frac{5b - 2r\Omega_1}{5a + 2r\Omega_2}.$$

The result obtained above, that the centre of the sphere may describe a parabola, enables us to explain a well-known phenomenon in billiards, which may be stated as follows:—

The angle at which the striker's ball goes off the ball aimed at, in order to make a cannon, seems to be less according as the distance of the third ball is greater. Let A be the striker's ball, B the ball first struck by A. If A be struck in the ordinary way, without side, it will have, when it strikes B, a rotation round a horizontal axis CR at right angles to SC, the line of original motion. This rotation will continue after impact round the same axis. Suppose the motion of translation after impact to be in the direction CT, then CR not coinciding with CR', the horizontal line perpendicular to CT, the path described by the ball is the parabolic arc CPP'. Hence, if P' be more remote than P, PCS' is greater than $P'CS'$. It is well known that a skilful billiard player can make a ball describe a very marked curve. This is done by an impulse having a *vertical component* which imparts a rotation round the line of original translation of the centre.

If the original impulse be horizontal it produces no moment round a line through the centre parallel to itself, and this latter being the original line of translation of the centre, there can be no rotation round it; hence in this case the ball must move in a straight line.



41. A sphere rolls along a rough horizontal plane. Taking into account the couple of rolling friction, determine the forces brought into play and the path described, the motion being pure rolling.

The equations of motion are

$$M\frac{d^2x}{dt^2} = X, \quad M\frac{d^2y}{dt^2} = Y,$$

$$\tfrac{2}{5}Mr^2\frac{d\omega_1}{dt} = rY - fMg\frac{\omega_1}{\omega}, \; \tfrac{2}{5}Mr^2\frac{d\omega_2}{dt} = -rX - fMg\frac{\omega_2}{\omega}, \text{ where } \omega = \sqrt{\omega_1{}^2 + \omega_2{}^2},$$

with the conditions $\quad \dfrac{dx}{dt} - r\omega_2 = 0, \quad \dfrac{dy}{dt} + r\omega_1 = 0.$

whence $\quad X = -\tfrac{5}{7}\dfrac{f}{r}Mg\dfrac{\omega_2}{\omega}, \quad Y = \tfrac{5}{7}\dfrac{f}{r}Mg\dfrac{\omega_1}{\omega}, \quad \dfrac{Y}{X} = \dfrac{-\omega_1}{\omega_2} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$

Hence the path is a straight line. Multiplying the third equation of motion by ω_1, the fourth by ω_2, and adding, we have

$$\tfrac{2}{5}Mr^2\omega\frac{d\omega}{dt} = -\tfrac{2}{7}fMg\omega,$$

whence
$$r\omega = -\tfrac{5}{7}\frac{fg}{r}t + r\Omega,$$

Ω being the initial angular velocity.

The sphere will come to rest when $t = \dfrac{7r^2\Omega}{5fg}$.

Again multiplying the third equation of motion by ω_2, the fourth by ω_1, and subtracting, we have $\omega_2 d\omega_1 - \omega_1 d\omega_2 = 0$. Hence $\dfrac{\omega_1}{\omega_2} = \text{constant} = -\tan\alpha$, where α is the angle which the path makes with the axis of x.

42. A sphere is projected obliquely down a rough inclined plane, the motion being pure rolling; determine the friction brought into play, and the path, neglecting the couple of rolling friction.

Take as axis of x the intersection of the inclined plane with a vertical plane at right angles thereto.

The equations of motion are

$$M\frac{d^2x}{dt^2} = X + Mg\sin i, \quad M\frac{d^2y}{dt^2} = Y, \quad \tfrac{2}{5}Mr^2\frac{d\omega_1}{dt} = rY, \quad \tfrac{2}{5}Mr^2\frac{d\omega_2}{dt} = -rX,$$

with the conditions $\quad \dfrac{dx}{dt} - r\omega_2 = 0, \quad \dfrac{dy}{dt} + r\omega_1 = 0;$

whence $\quad Y = 0, \quad X = -\tfrac{2}{7}Mg\sin i.$

The whole force, therefore, is $\tfrac{5}{7}Mg\sin i$ parallel to the axis of x, and the centre of the sphere being acted on by a constant force parallel to a fixed direction, describes a parabola. Since

$$\frac{\dfrac{dx}{dt}}{\dfrac{dy}{dt}} = -\frac{\omega_2}{\omega_1},$$

the instantaneous axis of rotation is at right angles to the tangent to the path of the point of contact on the inclined plane. This is otherwise immediately obvious, since the motion is pure rolling.

43. A sphere is projected in any way along an imperfectly rough inclined plane. Investigate the motion.

(This investigation, with some slight modifications, is taken from Routh, *Rigid Dynamics.*)

As in Ex. 42, the equations of motion are

$$M\frac{d^2x}{dt^2} = X + Mg\sin i, \quad M\frac{d^2y}{dt^2} = Y,$$

$$\tfrac{2}{5}Mr^2\frac{d\omega_1}{dt} = rY, \quad \tfrac{2}{5}Mr^2\frac{d\omega_2}{dt} = -rX,$$

whence, eliminating X and Y, we obtain, on integrating,

$$\frac{dx}{dt} + \tfrac{2}{5}r\omega_2 = gt\sin i + \alpha + \tfrac{2}{5}r\Omega_2,$$

$$\frac{dy}{dt} - \tfrac{2}{5}r\omega_1 = \beta - \tfrac{2}{5}r\Omega_1,$$

where α, β, Ω_1, and Ω_2 are the initial values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, ω_1, and ω_2.

Again, if u be the velocity at any instant of that point of the sphere which is in contact with the plane, and θ the angle which its direction makes with the axis of x,

$$u\cos\theta = \frac{dx}{dt} - r\omega_2, \quad u\sin\theta = \frac{dy}{dt} + r\omega_1.$$

Differentiating, substituting for $\frac{d^2x}{dt^2}$, &c., from the equations of motion, putting for X and Y the values which they take as long as there is slipping, viz., $-\mu Mg\cos i\cos\theta$ and $-\mu Mg\cos i\sin\theta$, and solving the resulting equations for $\frac{du}{dt}$ and $u\frac{d\theta}{dt}$, we have

$$\frac{du}{dt} = g\sin i\cos\theta - \tfrac{7}{2}\mu g\cos i, \quad u\frac{d\theta}{dt} = -g\sin i\sin\theta.$$

Hence, if $\frac{7}{2}\mu\cot i = n$, we obtain, by integration, $u\sin\theta = K_1(\tan\frac{1}{2}\theta)^n$. Substituting the value given by this equation for u in the equation for $u\frac{d\theta}{dt}$, and integrating, we have

$$\frac{(\tan\frac{1}{2}\theta)^{n+1}}{n+1} + \frac{(\tan\frac{1}{2}\theta)^{n-1}}{n-1} = K_2 - \frac{2g\sin i}{K_1}t.$$

K_2 is determined from the initial value of θ, and K_1 from the initial values of θ and u. These latter are given by the equations

$$u_0\cos\theta_0 = \alpha - r\Omega_2, \quad u_0\sin\theta_0 = \beta + r\Omega_1,$$

then u and θ being known, $\frac{dx}{dt}$, $\frac{dy}{dt}$, ω_1, and ω_2 can be determined.

If n or $\frac{7}{2}\mu \cot i > 1$, u and θ become continually less until they vanish together. Pure rolling then begins at a time t_0, which is given by the equation $t_0 = \dfrac{K_1 K_2}{2g \sin i}$. After pure rolling begins the values of $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \omega_1$, and ω_2, at any time, can be obtained from the combination of the equations of motion with the equations

$$\frac{dx}{dt} - r\omega_2 = 0, \quad \frac{dy}{dt} + r\omega_1 = 0.$$

If $n < 1$, θ, though constantly approaching zero, as appears from the expression for $u\dfrac{d\theta}{dt}$, will not vanish in any finite time, and u tends to increase without limit.

If $u_0 = 0$, the problem is at starting reduced to that of Ex. 42. The force of friction requisite for pure rolling is then $\frac{2}{7}Mg \sin i$. Hence, if

$$\tfrac{2}{7}Mg \sin i < \mu' Mg \cos i, \quad \text{or} \quad \tfrac{7}{2}\mu' \cot i > 1,$$

where μ' is the coefficient of *statical* friction, pure rolling will commence and continue. If $\frac{7}{2}\mu' \cot i < 1$, slipping will begin at once and never cease.

44. A solid of revolution, having a great angular velocity round its axis, and terminated by a spherical surface of small radius, is placed, with its axis inclined to the vertical, on a rough horizontal plane. The moment of inertia round the axis of revolution being not less than that round an axis perpendicular thereto, and the distance of the centre of inertia from the lower end being considerable, show that after some time the axis of revolution will become vertical. (Jellett, *Theory of Friction*, Chapter VIII.)

Let the axis of z through the centre of inertia O be vertical, and let OC be the axis of revolution, which must pass through S, the centre of the terminating spherical surface. Accordingly the point of contact T lies in the plane ZOC.

The forces acting on the body are its own weight, and the resultant of the normal reaction and friction at T. The latter may be resolved into two, one along TZ', the other at right angles to the plane ZOC. Calling this latter component F, and putting

$$TS = a, \quad SO = b, \quad ZOC = \theta,$$

the moments of the applied forces round OZ and OC are

$$Fb \sin\theta, \text{ and } -Fa \sin\theta.$$

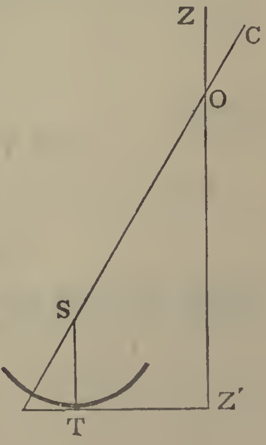
Now (Equations (20), p. 330, and Ex. 29, p. 317)

$$H_3 = A(-\omega_1 \sin\theta \cos\phi + \omega_2 \sin\theta \sin\phi) + C\omega_3 \cos\theta$$

$$= A \sin^2\theta \frac{d\psi}{dt} + C\omega_3 \cos\theta. \quad \text{(Art. 238.)}$$

Hence (Art. 254),

$$\frac{d}{dt}\left(A \sin^2\theta \frac{d\psi}{dt} + C\omega_3 \cos\theta\right) = Fb \sin\theta\,;$$



also (Art. 241), since $B = A$, $C\dfrac{d\omega_3}{dt} = -Fa\sin\theta$; therefore

$$\frac{d}{dt}\left(A\sin^2\theta\frac{d\psi}{dt} + C\omega_3\cos\theta\right) = -\frac{b}{a}\,C\frac{d\omega_3}{dt}.$$

Now, as ω_1 and ω_2 are each zero initially, $\dfrac{d\psi}{dt} = 0$ initially (Art. 238, (8)). Hence, if Ω and θ_0 are the initial values of ω_3 and θ,

$$A\sin^2\theta\frac{d\psi}{dt} + C\omega_3\left(\frac{b}{a} + \cos\theta\right) - C\Omega\left(\frac{b}{a} + \cos\theta_0\right) = 0.$$

As the force F is constantly diminishing ω_3, after some time ω_3 must become equal to

$$\Omega\,\frac{b + a\cos\theta_0}{b + a}.$$

When this happens $\theta = 0$. For, putting $n = \dfrac{b}{a}$, and substituting in the previous equation the value just obtained for ω_3, we get

$$(1 - \cos\theta)\left\{2A\cos^2\tfrac{1}{2}\theta\frac{d\psi}{dt} - C\Omega\frac{n + \cos\theta_0}{n+1}\right\} = 0;$$

but as n is greater than 1, $\dfrac{d\psi}{dt}$ inconsiderable as compared with Ω, and C not less than A, the second factor of the above expression cannot vanish, and therefore we must have $\theta = 0$.

The result obtained here may be regarded as holding good in the case of a top.

45. A body having a fixed point is set in motion by a given impulse, determine the direction in which a fixed axis through the point should be placed in order that the vis viva imparted should be a maximum.

I being the moment of inertia round the axis, and ω the angular velocity imparted by the impulse,

$$I\omega = \underset{\cdot}{L}\frac{\omega_1}{\omega} + \underset{\cdot}{M}\frac{\omega_2}{\omega} + \underset{\cdot}{N}\frac{\omega_3}{\omega};$$

therefore

$$\underset{\cdot}{L}\omega_1 + \underset{\cdot}{M}\omega_2 + \underset{\cdot}{N}\omega_3 = I\omega^2 = A\omega_1^2 + B\omega_2^2 + C\omega_3^2.$$

If $A\omega_1^2 + B\omega_2^2 + C\omega_3^2$ be a maximum subject to the condition above,

$$A\omega_1 = \lambda\underset{\cdot}{L}, \quad B\omega_2 = \lambda\underset{\cdot}{M}, \quad C\omega_3 = \lambda\underset{\cdot}{N}.$$

Substituting in the equation of condition we get $\lambda = 1$. Hence the axis required, if it exist, is determined by the equations

$$\omega_1 = \frac{\underset{\cdot}{L}}{A}, \quad \omega_2 = \frac{\underset{\cdot}{M}}{B}, \quad \omega_3 = \frac{\underset{\cdot}{N}}{C}.$$

To determine whether the vis viva T is really a maximum, put $\omega_1 + \alpha$ for ω_1, $\omega_2 + \beta$ for ω_2, $\omega_3 + \gamma$ for ω_3, then

$$T' - T = 2\,(A\omega_1\alpha + B\omega_2\beta + C\omega_3\gamma) + A\alpha^2 + B\beta^2 + C\gamma^2,$$

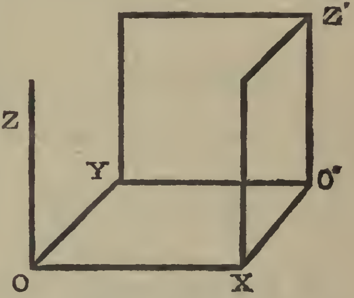
but, since $T' = \underset{\cdot}{L}(\omega_1 + \alpha) + \underset{\cdot}{M}(\omega_2 + \beta) + \underset{\cdot}{N}(\omega_3 + \gamma) = T + \underset{\cdot}{L}\alpha + \underset{\cdot}{M}\beta + \underset{\cdot}{N}\gamma$,

we have

$$A\omega_1\alpha + B\omega_2\beta + C\omega_3\gamma + A\alpha^2 + B\beta^2 + C\gamma^2 = 0,$$

therefore, $A\omega_1\alpha + B\omega_2\beta + C\omega_3\gamma$, and $T' - T$ are each negative, and T a maximum.

46. A sphere, having no original velocity of rotation, impinges successively against two perfectly rough vertical walls at right angles to each other, the points of impact being so near the intersection of the walls that the action of gravity between the two impacts may be neglected; determine the magnitude and direction of the velocity of the centre of the sphere after the second impact.



Take as axes of ξ, η, ζ, three lines through the centre of the sphere parallel to the intersections of the walls with the plane of the horizon and with each other. Let the components of the velocity of the sphere before the first impact be V_1, V_2, V_3, and let the sphere impinge first against the plane $YO'Z'$.

At the first impact the coordinates ξ, η, ζ of the point of contact are given by the equations

$$\xi = 0, \quad \eta = a, \quad \zeta = 0.$$

Hence (Art 235, (3)),

$$\frac{dx}{dt} = \frac{d\bar{x}}{dt} - a\omega_3, \quad \frac{dy}{dt} = \frac{d\bar{y}}{dt}, \quad \frac{dz}{dt} = \frac{d\bar{z}}{dt} + a\omega_1,$$

where x, y, z are the coordinates of the point of contact referred to axes meeting at O.

Again, $\underset{\cdot}{X}$, $\underset{\cdot}{Z}$, and R being the impulses due to friction and to the normal reaction during the first period of the collision (Art. 255).

$$m\frac{d\bar{x}}{dt} = mV_1 + \underset{\cdot}{X}, \quad m\frac{d\bar{y}}{dt} = mV_2 + R, \quad m\frac{d\bar{z}}{dt} = mV_3 + Z,$$

$$\tfrac{2}{5}ma^2\omega_1 = a\underset{\cdot}{Z}, \quad \omega_2 = 0, \quad \tfrac{2}{5}ma^2\omega_3 = -a\underset{\cdot}{X},$$

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 0.$$

Hence $$\frac{d\bar{x}}{dt} = \tfrac{5}{7}V_1, \quad \frac{d\bar{y}}{dt} = 0, \quad \frac{d\bar{z}}{dt} = \tfrac{5}{7}V_3,$$

$$a\omega_1 = -\tfrac{5}{7}V_3, \quad a\omega_2 = 0, \quad a\omega_3 = \tfrac{5}{7}V_1.$$

In the second period of the collision five of the six velocities above remain unaltered, but $\frac{d\bar{y}}{dt}$ becomes $-eV_2$.

At the second impact the coordinates of the point of contact are

$$\xi = a, \quad \eta = 0, \quad \zeta = 0.$$

Hence

$$\frac{dx}{dt} = \frac{d\bar{x}}{dt}, \quad \frac{dy}{dt} = \frac{d\bar{y}}{dt} + a\omega_3, \quad \frac{dz}{dt} = \frac{d\bar{z}}{dt} - a\omega_2.$$

Also
$$\left(\frac{d\bar{x}}{dt}\right)_0 = \tfrac{5}{7} V_1, \quad \left(\frac{d\bar{y}}{dt}\right)_0 = -eV_2, \quad \left(\frac{d\bar{z}}{dt}\right)_0 = \tfrac{5}{7} V_3,$$

$$a\Omega_1 = -\tfrac{5}{7} V_3, \quad a\Omega_2 = 0, \quad a\Omega_3 = \tfrac{5}{7} V_1.$$

Proceeding as before we obtain

$$a\omega_1 = -\tfrac{5}{7} V_3, \quad a\omega_2 = (\tfrac{5}{7})^2 V_3, \quad a\omega_3 = \tfrac{5}{7} e V_2 + \tfrac{2}{5} (\tfrac{5}{7})^2 V_1,$$

$$\frac{d\bar{x}}{dt} = -\tfrac{5}{7} e V_1, \quad \frac{d\bar{y}}{dt} = -\tfrac{5}{7} e V_2 - \tfrac{2}{5} \cdot (\tfrac{5}{7})^2 V_1, \quad \frac{d\bar{z}}{dt} = (\tfrac{5}{7})^2 V_3.$$

47. In the last example, if the walls be not perfectly rough, determine the final velocities of translation and rotation (see Art. 255).

We first treat the question as before and obtain, as in the last example, at the first impact,

$$\frac{1}{m^2}(X^2 + Z^2) = \tfrac{4}{49}(V_1^2 + V_3^2); \quad \text{if then } \mu(1+e) V_2 > \tfrac{2}{7}\sqrt{(V_1^2 + V_3^2)},$$

we may assume there is no slipping (Art. 256); but if this condition is not fulfilled slipping takes place, and the maximum amount of friction is exerted. In this case at the end of the first impact,

$$\frac{d\bar{x}}{dt} = V_1 - \mu(1+e) V_2 \cos\alpha, \quad \frac{d\bar{y}}{dt} = -eV_2, \quad \frac{d\bar{z}}{dt} = V_3 - \mu(1+e) V_2 \sin\alpha,$$

$$a\omega_1 = -\tfrac{5}{2}\mu(1+e) V_2 \sin\alpha, \quad \omega_2 = 0, \quad a\omega_3 = \tfrac{5}{2}\mu(1+e) V_2 \cos\alpha,$$

where $\tan\alpha = \frac{V_3}{V_1}$.

The values of $\frac{d\bar{x}}{dt}$, &c., ω_1, &c., at the end of the first impact are the values of $\left(\frac{d\bar{x}}{dt}\right)_0$, &c., Ω_1, &c., at the second impact. If slipping takes place during the whole of the second impact, we have finally

$$\frac{d\bar{x}}{dt} = -e\left\{V_1 - \mu(1+e)\,V_2\cos\alpha\right\},$$

$$\frac{d\bar{y}}{dt} = -eV_2 - \mu(1+e)\left\{V_1 - \mu(1+e)\,V_2\cos\alpha\right\}\cos\beta,$$

$$\frac{d\bar{z}}{dt} = V_3 - \mu(1+e)\left\{V_2\sin\alpha + V_1\sin\beta - \mu(1+e)\,V_2\cos\alpha\sin\beta\right\},$$

$$a\pi_1 = 0,\quad a\pi_2 = \tfrac{5}{2}\,\mu(1+e)\left\{V_1 - \mu(1+e)\,V_2\cos\alpha\right\}\sin\beta,$$

$$a\pi_3 = -\tfrac{5}{2}\,\mu(1+e)\left\{V_1 - \mu(1+e)\,V_2\cos\alpha\right\}\cos\beta,$$

$$a\omega_1 = -\tfrac{5}{2}\,\mu(1+e)\,V_2\sin\alpha,$$

$$a\omega_2 = \tfrac{5}{2}\,\mu(1+e)\left\{V_1 - \mu(1+e)\,V_2\cos\alpha\right\}\sin\beta,$$

$$a\omega_3 = \tfrac{5}{2}\,\mu(1+e)\left\{V_2\cos\alpha\,[1 + \mu(1+e)\cos\beta] - V_1\cos\beta\right\},$$

where
$$\tan\beta = \frac{2\left\{V_3 - \mu(1+e)\,V_2\sin\alpha\right\}}{5\mu(1+e)\,V_2\cos\alpha - 2eV_2}.$$

48. A body rests with a plane face on an imperfectly rough horizontal plane. The centre of inertia of the body is vertically over the centre of inertia of the face and very near it, the connecting line being a principal axis at the former point. The form of the face is such, that its radii of gyration about all lines in it passing through its centre of inertia are equal. The body is projected with an initial velocity of translation U, and an initial very small angular velocity Ω round a vertical axis through its centre of inertia: determine the motion.

Take the initial direction of translation and a horizontal line at right angles thereto for axes of x and y. Let u and v be the components of the velocity of the centre of inertia of the body at any time, and ω the angular velocity. Then, x and y being the coordinates of any point of the body, and ξ and η its coordinates referred to parallel axes through the centre of inertia,

$$\frac{dx}{dt} = u - \eta\omega,\quad \frac{dy}{dt} = v + \xi\omega.$$

If F be the magnitude of the whole force of friction at any point, its components X and Y are given by the equations

$$X = -F\frac{u - \eta\omega}{\sqrt{\{(u-\eta\omega)^2 + (v+\xi\omega)^2\}}} = -F, \qquad q.p.$$

$$Y = -F\frac{v + \xi\omega}{\sqrt{\{(u-\eta\omega)^2 + (v+\xi\omega)^2\}}} = -F\frac{v+\xi\omega}{u}. \qquad q.p.$$

since v, $\xi\omega$, and $\eta\omega$ are small compared with u.

Again, if S be the area of the plane face, the magnitude of the normal reaction of the horizontal plane on an element of the face is equal to $\phi(\xi, \eta)\,dS$, whence $F = \mu\,\phi(\xi, \eta)\,dS$, and, since $\bar{z} =$ constant, $\int\phi(\xi, \eta)\,dS = mg$, where m is the mass of the body. Also equations (22), of Art. 254, give $L = 0$, $M = 0$, since $\omega_x = 0$, $\omega_y = 0$, $i = 0$, $j = 0$.

If a be the distance of the centre of inertia of the body from the plane face, and $\phi(\xi, \eta) = R$,

$$M = \mu a\int RdS - \int R\xi dS;$$

therefore
$$\int R\xi dS = \mu mga.$$

Assume $R = K + \epsilon\Lambda$, where K and ϵ are constants, then

$$\mu mga = K\int\xi dS + \epsilon\int\Lambda\xi ds, \quad \text{but} \quad \int\xi dS = 0;$$

therefore ϵ must be small, also

$$mg = KS + \epsilon\int\Lambda dS.$$

Again
$$L = \int R\eta dS - \mu a\int\frac{v+\xi\omega}{u}RdS,$$

and, since the second member of L is zero, $q.p.$, we have $\int R\eta dS = 0$. Hence the resultant normal reaction passes through a point on the axis of x.

To determine the motion of the centre of inertia,

$$m\frac{du}{dt} = \Sigma X = -\mu\int RdS = -\mu mg;$$

therefore
$$u = U - \mu gt.$$

Again
$$m\frac{dv}{dt} = \Sigma Y = -\mu\frac{v}{u}\int RdS - \mu\frac{\omega}{u}\int R\xi dS = -\mu mg\frac{v}{u}, \qquad q.p.$$

hence $v = cu$; and since $v = 0$ when $u = U$, $c = 0$, therefore $v = 0$.

To find the angular velocity,

$$mk^2\frac{d\omega}{dt} = -\mu\frac{\omega}{u}\int R\xi^2 dS = -\mu K\frac{\omega}{u}\int\xi^2 dS; \qquad q.p.$$

but γ being the radius of gyration of the plane face, $\int \xi^2 dS = S\gamma^2$, and

$$mk^2 \frac{d\omega}{dt} = -\mu mg\gamma^2 \frac{\omega}{u}; \qquad q.p.$$

therefore $$\omega = \Omega \left(\frac{u}{U}\right)^{\frac{\gamma^2}{k^2}}.$$

49. A rigid body is moving in any manner; one point is suddenly arrested: determine the impulse exerted on the body.

Let u, v, w be the components of the velocity of the point immediately before it is arrested, x, y, z its coordinates, and $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$ the components of the impulse, the axes being the principal axes of the body at the centre of inertia, then $\underset{\cdot}{X}$ is given by the equation

$$-\left\{\frac{ABC}{\mathfrak{M}} + A(B+C)x^2 + B(C+A)y^2 + C(A+B)z^2 + \mathfrak{M}Ir^4\right\} \underset{\cdot}{X}$$

$$= \{ABC + \mathfrak{M}[A(B+C)x^2 + BC(y^2+z^2) + \mathfrak{M}Ir^2x^2]\}\, u$$

$$+ \mathfrak{M}(AB + \mathfrak{M}Ir^2)\, xyv + \mathfrak{M}(AC + \mathfrak{M}Ir^2)\, xzw,$$

where I is the moment of inertia of the body round the line joining the arrested point to the centre of inertia, r the distance between these points, and A, B, C, the principal moments of inertia of the body.

CHAPTER XI.

SMALL OSCILLATIONS.*

257. **Introductory Considerations.**—When a material system in equilibrium under the action of any forces is slightly disturbed, the several points of the system in many cases tend to return to their original positions. In such cases, if the distance of each point from its initial position remains during the motion very small as compared with the other magnitudes on which the motion depends, the system performs *small oscillations.*

Some cases of small oscillations have been already considered in Articles 100 and 188. The simplification of the problem in the case of small oscillations has been exemplified in the articles referred to, and consists in neglecting the squares and higher powers of small quantities.

It is assumed that the forces which act at the different points of the system are functions of the coordinates of those points, and that the restraints and mutual connexions can be expressed by means of equations between the coordinates.

In virtue of these equations the coordinates of the points of the system are functions of i independent variables, and these again are at any time functions of their initial values, and of the increments resulting from the motion. If the system perform small oscillations the increments of the i variables are all small quantities whose squares and higher powers may be neglected.

Hence the equations of motion involve only the first powers of the variables and of their differential coefficients. In other words, they form a system of linear differential equations with constant coefficients.

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* For a great part of the materials of this chapter we are indebted to Mr. Townsend, who kindly placed at our disposal his Lectures on Small Oscillations, delivered to his University Class.

258. **Equations of Motion.**—The fundamental equation of Dynamics, which holds good for every system, is supplied by D'Alembert's Principle (Art. 201). To obtain from it the equations of motion we equate to cipher the coefficient of every independent variation which the equation contains.

In the present case, x, y, z, &c., are supposed to be functions of the i variables θ, ϕ, ψ, &c., whose variations are each entirely arbitrary. We have therefore to express $\frac{d^2x}{dt^2}$, X, δx, &c., in terms of θ, ϕ, ψ, &c., and then equate to cipher the coefficients of $\delta\theta$, $\delta\phi$, &c. If θ_0, ϕ_0, ψ_0, &c., be the initial values of θ, ϕ, ψ, &c., and u, v, w, &c., their increments at any time, we have $\theta = \theta_0 + u$, $\phi = \phi_0 + v$, &c.; and $\delta\theta = \delta u$, $\delta\phi = \delta v$, &c.

Equation (1), Art. 201, may be written

$$\Sigma m\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z\right) = \Sigma\left(X\delta x + Y\delta y + Z\delta z\right).$$

It will be shown in the next chapter that when the forces are functions of the coordinates, and are such as exist in nature, $\Sigma(Xdx + Ydy + Zdz)$ must be the exact differential of some function V of the coordinates (see Arts. 122, 123). In this case we may write

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = \delta V.$$

Since V is a function of x, y, z, &c., which again are functions of θ, ϕ, ψ, &c., we have

$$V = F(\theta, \phi, \psi, \text{&c.}) = F(\theta_0 + u, \phi_0 + v, \psi_0 + w, \text{&c.})$$

$$= V_0 + h_1 u + h_2 v + h_3 w + \text{&c.}$$

$$-\tfrac{1}{2}\{p_{11}u^2 + p_{22}v^2 + p_{33}w^2 + \text{&c.} + 2p_{12}uv + 2p_{13}uw + 2p_{23}vw + \text{&c.}\},$$

as the higher powers of u, v, w, &c., may be neglected.

Hence

$$\delta V = \{h_1 - (p_{11}u + p_{12}v + p_{13}w + \&c.)\}\,\delta u$$
$$+ \{h_2 - (p_{12}u + p_{22}v + p_{23}w + \&c.)\}\,\delta v$$
$$+ \{h_3 - (p_{13}u + p_{23}v + p_{33}w + \&c.)\}\,\delta w + \&c.$$

Again, $x = f(\theta, \phi, \psi, \&c.) = f(\theta_0 + u, \phi_0 + v, \psi_0 + w, \&c.)$

$$= x_0 + a_x u + b_x v + c_x w + \&c.$$

In like manner,

$$y = y_0 + a_y u + b_y v + c_y w + \&c.,$$
$$z = z_0 + a_z u + b_z v + c_z w + \&c.,$$
$$\&c., \qquad \&c., \qquad \&c.$$

Adopting Newton's notation (Art. 21) we get

$$\ddot{x} = a_x\ddot{u} + b_x\ddot{v} + c_x\ddot{w} + \&c., \quad \delta x = a_x\delta u + b_x\delta v + c_x\delta w + \&c.,$$
$$\&c., \qquad\qquad \&c.$$

Hence $\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z)$

$$= \Sigma m(a_x\ddot{u} + b_x\ddot{v} + c_x\ddot{w} + \&c.)(a_x\delta u + b_x\delta v + c_x\delta w + \&c.)$$
$$+ \Sigma m(a_y\ddot{u} + b_y\ddot{v} + c_y\ddot{w} + \&c.)(a_y\delta u + b_y\delta v + c_y\delta w + \&c.)$$
$$+ \Sigma m(a_z\ddot{u} + b_z\ddot{v} + c_z\ddot{w} + \&c.)(a_z\delta u + b_z\delta v + c_z\delta w + \&c.)$$
$$= (f_{11}\ddot{u} + f_{12}\ddot{v} + f_{13}\ddot{w} + \&c.)\,\delta u + (f_{12}\ddot{u} + f_{22}\ddot{v} + f_{23}\ddot{w} + \&c.)\,\delta v$$
$$+ (f_{13}\ddot{u} + f_{23}\ddot{v} + f_{33}\ddot{w} + \&c.)\,\delta w + \&c.,$$

where $f_{11} = \Sigma m(a_x^2 + a_y^2 + a_z^2), \quad f_{12} = \Sigma m(a_x b_x + a_y b_y + a_z b_z),$

$$f_{22} = \Sigma m(b_x^2 + b_y^2 + b_z^2), \quad f_{23} = \Sigma m(b_x c_x + b_y c_y + b_z c_z),$$
$$f_{34} = \Sigma m(c_x d_x + c_y d_y + c_z d_z), \qquad \&c.$$

Equating now the coefficients of the independent variations δu, δv, &c. in $\Sigma m(\ddot{x}\delta x+\ddot{y}\delta y+\ddot{z}\delta z)$, and in δV, we obtain the system of linear differential equations

$$\left.\begin{array}{l} f_{11}\ddot{u}+f_{12}\ddot{v}+f_{13}\ddot{w}+\&c.+p_{11}u+p_{12}v+p_{13}w+\&c.=h_1 \\ f_{12}\ddot{u}+f_{22}\ddot{v}+f_{23}\ddot{w}+\&c.+p_{12}u+p_{22}v+p_{23}w+\&c.=h_2 \\ f_{13}\ddot{u}+f_{23}\ddot{v}+f_{33}w+\&c.+p_{13}u+p_{23}v+p_{33}w+\&c.=h_3 \\ \qquad\qquad\qquad \&c. \qquad\qquad\qquad\qquad\qquad\qquad =\&c. \end{array}\right\}. \quad (1)$$

When h_1, h_2, &c., are each zero, the assumption of $u=0$, $v=0$, &c., satisfies the equations of motion, and the initial position is therefore one of equilibrium. Conversely, if the initial position be one of equilibrium, $\Sigma(X\delta x+Y\delta y+Z\delta z)$, or δV is zero when $u=0$, $v=0$, &c., for all values of δu, δv, &c. But when $u=0$, $v=0$, &c., $\delta V=h_1\delta u+h_2\delta v+\&c.$; therefore $h_1=0$, $h_2=0$, &c. In our subsequent discussion we shall assume that the initial position is one of equilibrium, and therefore that $h_1=0$, $h_2=0$, &c.

259. **Solution of Equations of Motion.**—If we assume

$$u=ka\sin(t\sqrt{\lambda}+\chi),\quad v=kb\sin(t\sqrt{\lambda}+\chi),\ \&c.,$$

and substitute in the equations of motion, we obtain the i equations

$$\left.\begin{array}{l} (f_{11}\lambda-p_{11})a+(f_{12}\lambda-p_{12})b+(f_{13}\lambda-p_{13})c+\&c.=0 \\ (f_{12}\lambda-p_{12})a+(f_{22}\lambda-p_{22})b+(f_{23}\lambda-p_{23})c+\&c.=0 \\ \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ (f_{1i}\lambda-p_{1i})a+(f_{2i}\lambda-p_{2i})b+(f_{3i}\lambda-p_{3i})c+\&c.=0 \end{array}\right\}, \quad (2)$$

which can be satisfied by the ratios of the i determinable constants a, b, &c.; k and χ remaining arbitrary, provided λ is a root of the symmetrical determinant equation

$$\begin{vmatrix} f_{11}\lambda - p_{11}, & f_{12}\lambda - p_{12}, & . & . & . & f_{1i}\lambda - p_{1i} \\ f_{12}\lambda - p_{12}, & f_{22}\lambda - p_{22}, & . & . & . & . \\ . & . & . & . & . & . \\ f_{1i}\lambda - p_{1i}, & . & . & . & . & f_{ii}\lambda - p_{ii} \end{vmatrix} = 0. \quad (3)$$

This determinant we shall call Δ. It is usually termed the *harmonic determinant* of the motion.

If the roots of $\Delta = 0$ are all real and positive, the complete values of u, v, &c., are given by the equations

$$\left.\begin{aligned} u &= k_1 a_1 \sin(t\sqrt{\lambda_1} + \chi_1) + k_2 a_2 \sin(t\sqrt{\lambda_2} + \chi_2) \ldots + k_i a_i \sin(t\sqrt{\lambda_i} + \chi_i) \\ v &= k_1 b_1 \sin(t\sqrt{\lambda_1} + \chi_1) + k_2 b_2 \sin(t\sqrt{\lambda_2} + \chi_2) \ldots + k_i b_i \sin(t\sqrt{\lambda_i} + \chi_i) \\ &\text{\&c.,} \qquad\qquad \text{\&c.,} \qquad\qquad \text{\&c.} \end{aligned}\right\}, \quad (4)$$

where k_1, χ_1, k_2, χ_2, &c. are arbitrary constants, $2i$ in number, and a_1, b_1, c_1, &c. satisfy the i linear equations for a, b, c, &c. obtained by putting λ_1 for λ; a_2, b_2, c_2, &c. those obtained by putting λ_2 for λ; and so on.

If any root of $\Delta = 0$ be real and negative, the solution given above so far fails, as $\sqrt{\lambda_1}$ is imaginary when λ_1 is negative. In this case, instead of $k_1 a_1 \sin(t\sqrt{\lambda_1} + \chi_1)$ there will be in u a term of the form

$$a_1 \left\{ k_1 e^{t\sqrt{\mu_1}} + k_1' e^{-t\sqrt{\mu_1}} \right\},$$

where $\mu_1 = -\lambda_1$; and there will be corresponding terms in v, &c. In fact, if we substitute $k_1 a_1 e^{t\sqrt{\mu_1}}$ for u, $k_1 b_1 e^{t\sqrt{\mu}}$ for v, and so on, in the equations of motion, we get the system

$$\begin{aligned} (f_{11}\mu_1 + p_{11}) a_1 + (f_{12}\mu_1 + p_{12}) b_1 + \text{\&c.} &= 0, \\ (f_{12}\mu_1 + p_{12}) a_1 + \text{\&c.} &= 0, \\ \text{\&c.} &= 0, \end{aligned}$$

which can be satisfied by $a_1 : b_1 :$ &c., provided μ_1 is a root of

$$\begin{vmatrix} f_{11}\mu + p_{11}, & f_{12}\mu + p_{12}, & \cdot \\ f_{12}\mu + p_{12}, & \cdot\quad\cdot\quad\cdot & \cdot \\ \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot \end{vmatrix} = 0.$$

But this equation is what (3) becomes if we change λ into $-\mu$; therefore corresponding to every real negative value of λ there is a real positive value of μ.

260. **Lagrange's Equations.**—The vis viva of the motion at any time is $\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, and as in Art. 258, we get

$$\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \Sigma m(a_x \dot{u} + b_x \dot{v} + c_x \dot{w} + \&\text{c.})^2$$

$$+ \Sigma m(a_y \dot{u} + b_y \dot{v} + c_y \dot{w} + \&\text{c.})^2 + \Sigma m(a_z \dot{u} + b_z \dot{v} + c_z \dot{w} + \&\text{c.})^2.$$

Hence the kinetic energy T of the system is given by the equation

$$T = \tfrac{1}{2}\{f_{11}\dot{u}^2 + f_{22}\dot{v}^2 + f_{33}\dot{w}^2 + \&\text{c.} + 2f_{12}\dot{u}\dot{v} + 2f_{13}\dot{u}\dot{w} + \&\text{c.}\}$$

Differentiating with respect to $\dot{u}$, $\dot{v}$, $\dot{w}$, &c., we obtain

$$\frac{dT}{d\dot{u}} = f_{11}\dot{u} + f_{12}\dot{v} + f_{13}\dot{w} + \&\text{c.},$$

$$\frac{dT}{d\dot{v}} = f_{12}\dot{u} + f_{22}\dot{v} + f_{23}\dot{w} + \&\text{c.},$$

$$\&\text{c.} \quad = \quad \&\text{c.}$$

Hence also

$$\frac{d}{dt}\left(\frac{dT}{d\dot{u}}\right) = f_{11}\ddot{u} + f_{12}\ddot{v} + f_{13}\ddot{w} + \&\text{c.},$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{v}}\right) = f_{12}\ddot{u} + f_{22}\ddot{v} + f_{23}\ddot{w} + \&\text{c.},$$

$$\&\text{c.} \quad = \quad \&\text{c.}$$

Again, from the value for V, Art. 258, when $h_1 = 0$, $h_2 = 0$, &c., we have

$$-\frac{dV}{du} = p_{11}u + p_{12}v + p_{13}w + \&c.,$$

$$-\frac{dV}{dv} = p_{12}u + p_{22}v + p_{23}w + \&c.,$$

$$\&c. \quad = \quad \&c.$$

Accordingly the differential equations (1), Art. 258, are equivalent to

$$\frac{d}{dt}\left(\frac{dT}{d\dot{u}}\right) = \frac{dV}{du}, \quad \frac{d}{dt}\left(\frac{dT}{d\dot{v}}\right) = \frac{dV}{dv}, \ \&c. \tag{5}$$

These are a particular case of Lagrange's Equations of Motion in Generalized Coordinates, which will be proved in the next chapter.

It is plain that the equations for determining λ, a, b, c, &c., Art. 259, may be written in the form

$$\left(\frac{d^2T}{d\dot{u}^2}\lambda + \frac{d^2V}{du^2}\right)a + \left(\frac{d^2T}{d\dot{u}\,d\dot{v}}\lambda + \frac{d^2V}{du\,dv}\right)b + \&c. = 0,$$

$$\left(\frac{d^2T}{d\dot{u}\,d\dot{v}}\lambda + \frac{d^2V}{du\,dv}\right)a + \left(\frac{d^2T}{d\dot{v}^2}\lambda + \frac{d^2V}{dv^2}\right)b + \&c. = 0,$$

$$\&c. \qquad\qquad = 0.$$

If we suppose a, b, c, &c., substituted for $\dot{u}$, $\dot{v}$, $\dot{w}$, &c., in T; and for u, v, w, &c. in V; and denote the results of these substitutions by T' and V', these equations may be written

$$\frac{d}{da}(\lambda T' + V') = 0, \quad \frac{d}{db}(\lambda T' + V') = 0, \ \&c.$$

261. **Reality of the Roots of the Harmonic Equation.**—If we put the quadratic function of i variables

$$\tfrac{1}{2}\{f_{11}u^2 + f_{22}v^2 + f_{33}w^2 + 2f_{12}uv + 2f_{13}uw + 2f_{23}vw + \&c.\} = \mathfrak{T},$$

it appears from Art. 260, since the vis viva is essentially positive, that *the quantic* $\mathfrak{T}$ *is positive for all real values of the variables.*

Hence also we may conclude that $\mathfrak{T}$ can be transformed into the sum of i positive squares (*Differential Calculus*, note on Art. 163).

We shall now show that *the equation* $\Delta = 0$ *has all its roots real.*

The following proof of this important theorem is due to Thomson and Tait (*Natural Philosophy*, § 343 *k*) :—

We have seen (Art. 259) that if λ_1 be one of the roots of $\Delta = 0$, we have the equations

$$(f_{11}a_1 + f_{12}b_1 + f_{13}c_1 + \&c.)\lambda_1 = p_{11}a_1 + p_{12}b_1 + p_{13}c_1 + \&c.,$$

$$(f_{12}a_1 + f_{22}b_1 + f_{23}c_1 + \&c.)\lambda_1 = p_{12}a_1 + p_{22}b_1 + p_{23}c_1 + \&c.,$$

$$\&c. \quad = \quad \&c.$$

If we put

$$\tfrac{1}{2}(f_{11}a_1^2 + f_{22}b_1^2 + 2_{12}a_1b_1 + \&c.) = \mathfrak{T}(a)_1 = \mathfrak{T}_1,$$

$$\tfrac{1}{2}(p_{11}a_1^2 + p_{22}b_1^2 + 2p_{12}a_1b_1 + \&c.) = S(a)_1 = S_1,$$

these equations may be written

$$\lambda_1\frac{d\mathfrak{T}_1}{da_1} = \frac{dS_1}{da_1}, \quad \lambda_1\frac{d\mathfrak{T}_1}{db_1} = \frac{dS_1}{db_1}, \ \&c.$$

Multiplying the first by a_2, the second by b_2, &c., and adding, we get

$$\lambda_1\left(a_2\frac{d\mathfrak{T}_1}{da_1} + b_2\frac{d\mathfrak{T}_1}{db_1} + c_2\frac{d\mathfrak{T}_1}{dc_1} + \&c.\right) = a_2\frac{dS_1}{da_1} + b_2\frac{dS_1}{db_1} + c_2\frac{dS_1}{dc_1} + \&c.,$$

which may be written

$$\lambda_1\mathfrak{T}(a)_{12} = S(a)_{12}, \tag{6}$$

if we put

$$\mathfrak{T}(a)_{12} = a_2\frac{d\mathfrak{T}_1}{da_1} + b_2\frac{d\mathfrak{T}_1}{db_1} + \&c. = a_1\frac{d\mathfrak{T}_2}{da_2} + b_1\frac{d\mathfrak{T}_2}{db_2} + \&c.,$$

and assign a similar meaning to $S(a)_{12}$.

Suppose now, if possible, that the equation $\Delta = 0$ has a pair of conjugate roots λ_1 and λ_2, where $\lambda_1 = \mu + \nu\sqrt{-1}$, and $\lambda_2 = \mu - \nu\sqrt{-1}$.

The values of a_1, b_1, &c., corresponding to λ_1 being of the form $\alpha_1 + \alpha_2\sqrt{-1}$, $\beta_1 + \beta_2\sqrt{-1}$, &c., those of a_2, b_2, &c., corresponding to λ_2 are then of the form $\alpha_1 - \alpha_2\sqrt{-1}$, $\beta_1 - \beta_2\sqrt{-1}$, &c. We have, therefore,

$$\mathfrak{T}(a)_{12} = (\alpha_1 - \alpha_2\sqrt{-1})\{(\alpha_1 + \alpha_2\sqrt{-1})f_{11} + (\beta_1 + \beta_2\sqrt{-1})f_{12} + \&c.\}$$

$$+ (\beta_1 - \beta_2\sqrt{-1})\{(\alpha_1 + \alpha_2\sqrt{-1})f_{12} + (\beta_1 + \beta_2\sqrt{-1})f_{22} + \&c.\} + \&c.$$

$$= \alpha_1\frac{d\mathfrak{T}(a)_1}{d\alpha_1} + \beta_1\frac{d\mathfrak{T}(a)_1}{d\beta_1} + \&c. + \alpha_2\frac{d\mathfrak{T}(a)_2}{d\alpha_2} + \beta_2\frac{d\mathfrak{T}(a)_2}{d\beta_2} + \&c.$$

$$+ \sqrt{-1}\left(\alpha_1\frac{d\mathfrak{T}(a)_2}{d\alpha_2} + \beta_1\frac{d\mathfrak{T}(a)_2}{d\beta_2} + \&c.\right)$$

$$- \sqrt{-1}\left(\alpha_2\frac{d\mathfrak{T}(a)_1}{d\alpha_1} + \beta_2\frac{d\mathfrak{T}(a)_1}{d\beta_1} + \&c.\right)$$

$$= 2\mathfrak{T}(a)_1 + 2\mathfrak{T}(a)_2.$$

In like manner, $S(a)_{12} = 2S(a)_1 + 2S(a)_2$. Hence, by (6),

$$(\mu + \nu\sqrt{-1})\{\mathfrak{T}(a)_1 + \mathfrak{T}(a)_2\} - \{S(a)_1 + S(a)_2\} = 0.$$

Equating the real and imaginary parts separately to cipher, we have $\nu\{\mathfrak{T}(a)_1 + \mathfrak{T}(a)_2\} = 0$.

Now, as the quantic $\mathfrak{T}$ is positive for all real values of the variables, $\mathfrak{T}(a)_1$ and $\mathfrak{T}(a)_2$ must each be positive, and $\mathfrak{T}(a)_1 + \mathfrak{T}(a)_2$ cannot be zero; therefore $\nu = 0$, and λ_1 is real.

It is to be observed that, as Δ is the discriminant of $\lambda\mathfrak{T} - S$, where S is the quantic $\frac{1}{2}\{p_{11}u^2 + p_{22}v^2 + \&c. + 2p_{12}uv + \&c.\}$, the equation $\Delta = 0$ is unaltered by any linear transformation applied to the quantics S and $\mathfrak{T}$.

262. **Stability of the Motion.**—We shall suppose that $2\mathfrak{T}$ is of the form $u^2 + v^2 + w^2 + \&c.$, $2S$ being as before $p_{11}u^2 + p_{22}v^2 + 2p_{12}uv + \&c.$

Apply a linear transformation which will change $u^2 + v^2 + w^2 + \&c.$ into $\xi^2 + \eta^2 + \zeta^2 + \&c.$, and at the same time reduce S to its canonical form $P_1\xi^2 + P_2\eta^2 + P_3\zeta^2 + \&c.$ In order to do this by a real transformation assume

$$\begin{aligned} u &= u_1\xi + u_2\eta + u_3\zeta + \&c., \\ v &= v_1\xi + v_2\eta + v_3\zeta + \&c., \\ \&c. &= \&c., \end{aligned}$$

where the ratios $u_1 : v_1 : w_1 : \&c.$ are determined by the equations

$$\begin{aligned} p_{11}u_1 + p_{12}v_1 + p_{13}w_1 + \&c. &= \lambda_1 u_1, \\ p_{12}u_1 + p_{22}v_1 + p_{23}w_1 + \&c. &= \lambda_1 v_1, \\ \&c. &= \&c.; \end{aligned}$$

$u_2 : v_2 : w_2 : \&c.$ by the equations

$$\begin{aligned} p_{11}u_2 + p_{12}v_2 + p_{13}w_2 + \&c. &= \lambda_2 u_2, \\ p_{12}u_2 + p_{22}v_2 + p_{23}w_2 + \&c. &= \lambda_2 v_2, \\ \&c. &= \&c., \end{aligned}$$

and so on. The constants u_1, v_1, &c., u_2, v_2, &c., are obviously the values which a_1, b_1, &c., a_2, b_2, &c. (Art. 259), take in the special case in which $2\mathfrak{T}$ is of the form $u^2 + v^2 + w^2 + \&c.$

From the above equations, it follows that, when λ_1 and λ_2 are unequal, $u_1u_2 + v_1v_2 + w_1w_2 + \&c. = 0$, &c. For,

$$\lambda_1(u_1u_2 + v_1v_2 + w_1w_2 + \&c.) = u_2\left(\frac{dS}{du}\right)_1 + v_2\left(\frac{dS}{dv}\right)_1 + \&c.$$

$$= u_1\left(\frac{dS}{du}\right)_2 + v_1\left(\frac{dS}{dv}\right)_2 + \&c. = \lambda_2(u_1u_2 + v_1v_2 + \&c.).$$

We shall now show that S becomes of the required form, for

$$\frac{dS}{d\xi} = u_1\frac{dS}{du} + v_1\frac{dS}{dv} + \&c.$$

$$= u\left(\frac{dS}{du}\right)_1 + v\left(\frac{dS}{dv}\right)_1 + \&c. = \lambda_1(u_1u + v_1v + w_1w + \&c.)$$

$$= \lambda_1\{(u_1^2 + v_1^2 + w_1^2 + \&c.)\xi + (u_1u_2 + v_1v_2 + w_1w_2 + \&c.)\eta + \&c.\}$$

$$= \lambda_1(u_1^2 + v_1^2 + w_1^2 + \&c.)\xi.$$

It can be shown in a similar manner that

$$\frac{dS}{d\eta} = \lambda_2 (u_2^2 + v_2^2 + w_2^2 + \&c.)\, \eta, \text{ and so on.}$$

If then we assume, as is allowable,

$$u_1^2 + v_1^2 + w_1^2 + \&c. = 1, \quad u_2^2 + v_2^2 + w_2^2 + \&c. = 1, \&c.,$$

we have the equations

$$\frac{dS}{d\xi} = \lambda_1 \xi, \quad \frac{dS}{d\eta} = \lambda_2 \eta, \&c.$$

Hence $2S = \lambda_1 \xi^2 + \lambda_2 \eta^2 + \lambda_3 \zeta^2 + \&c.$, and at the same time

$$u^2 + v^2 + w^2 + \&c. = \xi^2 + \eta^2 + \zeta^2 + \&c.$$

As the transformation above is real, it follows that if S be reduced in *any way* to a form which contains only the squares of the variables, the signs of the coefficients of the different squares are the same as those of λ_1, λ_2, &c.

In order that every term in the general values of u, v, w, &c. should be periodic, it is necessary (Art. 259) that all the roots of $\Delta = 0$ should be positive. This condition, as we have just seen, is fulfilled if S be reducible to the sum of a number of squares with positive coefficients—in other words, if the function V (Art. 258) be *a maximum for the position of equilibrium of the system.* In this case, the system being slightly disturbed in any manner has no tendency to depart far from its initial position, and consequently u, v, &c. must remain small throughout the motion. The motion is then stable in its character whatever be the directions of the initial disturbances, and the initial position is one of stable equilibrium.

If V be not a maximum for the position of equilibrium of the system, that is, if some of the coefficients in S when reduced to the above form be negative, terms may occur in u, v, &c. which increase without limit with the time. In this case the initial position is not one of stable equilibrium, and the motion will not consist of small oscillations, unless the original disturbances be such that the arbitrary constants multiplying terms in u, &c. which increase without limit with the time, are each zero.

263. **Equal Roots.**—When the equation $\Delta = 0$ has equal roots, the solution in Art. 259 of the differential equations (1) seems to fail from not containing the requisite number of arbitrary constants; and we might suppose that terms containing t as a factor would occur in the values of u, v, &c., and therefore that a stable motion of oscillation would not take place for all possible small disturbances. Lagrange and Laplace both fell into this mistake, which was first pointed out by Dr. Routh.

The true theory depends upon the circumstance that when the equation $\Delta = 0$ has a double root λ_1, the system of i linear equations for determining a, b, &c. (Art. 259), can be satisfied by $(i - 2)$ of these quantities, the remaining two being arbitrary.

This may be proved as follows:—

If λ_1, λ_2 be two roots of the equation $\Delta = 0$, we have, by (6),

$$\lambda_1 \mathfrak{T}(a)_{12} = S(a)_{12} = \lambda_2 \mathfrak{T}(a)_{12};$$

hence
$$(\lambda_1 - \lambda_2)\, \mathfrak{T}(a)_{12} = 0.$$

When $\lambda_1 - \lambda_2$ becomes small without vanishing this equation shows that $\mathfrak{T}_{12} = 0$. Hence the quantity $\frac{2\mathfrak{T}_1}{a_1^2} - \frac{\mathfrak{T}_{12}}{a_1 a_2}$, where a_1 and a_2 are arbitrary, does not tend to become small along with $\lambda_1 - \lambda_2$, but remains equal to a quantity essentially positive, viz. $\frac{2\mathfrak{T}_1}{a_1^2}$. Now each term of $\frac{2\mathfrak{T}_1}{a_1^2} - \frac{\mathfrak{T}_{12}}{a_1 a_2}$ contains one of the quantities $\frac{b_1}{a_1} - \frac{b_2}{a_2}$, $\frac{c_1}{a_1} - \frac{c_2}{a_2}$, &c. as a factor, and is either of the form $k\{\phi(\lambda_1) - \phi(\lambda_2)\}$, or else of the form $k\psi(\lambda_1)\{\phi(\lambda_1) - \phi(\lambda_2)\}$; where k is an absolute constant, and $\phi(\lambda_1)$ and $\psi(\lambda_1)$ are each the value of some one of the ratios $\frac{b_1}{a_1}$, $\frac{c_1}{a_1}$, &c. Moreover, any one of the functions $\phi(\lambda_1)$ or $\psi(\lambda_1)$ is a fraction whose numerator and denominator are rational and integral functions of λ_1, and all these fractions have the same denominator, viz. $\frac{d\Delta_1}{dp_{11}}$. But if $\psi(\lambda_1)\{\phi(\lambda_1) - \phi(\lambda_2)\}$ does not become indefinitely small as λ_1 approaches λ_2, either $\phi(\lambda_1) - \phi(\lambda_2)$ does not become indefinitely small, or $\psi(\lambda_1)$ becomes indefinitely great. In the former case the first derived $\phi'(\lambda_1)$ must become indefinitely great, which cannot happen for any finite value of λ_1 except the denominator of $\phi(\lambda_1)$ be zero. Again, $\psi(\lambda_1)$ cannot be indefinitely great except its denominator be zero. Hence, if λ_1 be a double root of the equation $\Delta = 0$, we conclude that $\frac{d\Delta_1}{dp_{11}}$ must vanish.

Instead of taking a_1 and a_2 as arbitrary, we might have taken b_1 and b_2, or any other corresponding pair of constants. Hence all the minors of Δ_1 obtained by erasing the first row and first column, the second row and second column, &c., must vanish. Hence, when λ is a double root of the equation $\Delta = 0$, the system of i linear equations (2) of Art. 259, can be satisfied by $(i-2)$ of the quantities a, b, &c., the other two remaining arbitrary. It is on this account that, when two roots λ_1 and λ_2 become equal, the ratio $b_1 : a_1$ need not become equal to the ratio $b_2 : a_2$, &c.

We can now show, when two roots λ_1 and λ_2 are equal—(1) That the method already given of effecting the orthogonal transformation still holds good with a slight modification; (2) That it is unnecessary to introduce terms involving t as a factor into the solution of the differential equations for u, v, w, &c.

(1). As before (Art. 262),

$$u_1u_3 + v_1v_3 + w_1w_3 + \&c. = 0, \quad u_2u_3 + v_2v_3 + w_2w_3 + \&c. = 0,$$
$$\&c. = 0, \qquad\qquad \&c. = 0,$$

where $\frac{v_1}{u_1}$ and $\frac{v_2}{u_2}$ are in the present case as yet arbitrary.

The equation $u_1u_2 + v_1v_2 + w_1w_2 + \&c. = 0$ can now be satisfied by means of $v_2 : u_2$ while $v_1 : u_1$ still remains arbitrary.

Hence the transformation is complete; but one of the ratios which is determined in the case of unequal roots is arbitrary in the case of equal.

(2) When the roots are all unequal the general solution of the differential equations for u, v, w, &c., may be written

$$
\begin{aligned}
u &= a_1 \sin t\sqrt{\lambda_1} + a_1' \cos t\sqrt{\lambda_1} + a_2 \sin t\sqrt{\lambda_2} + a_2' \cos t\sqrt{\lambda_2} \\
&\qquad + a_3 \sin t\sqrt{\lambda_3} + a_3' \cos t\sqrt{\lambda_3} + \&c. \\
v &= b_1 \sin t\sqrt{\lambda_1} + b_1' \cos t\sqrt{\lambda_1} + b_2 \sin t\sqrt{\lambda_2} + b_2' \cos t\sqrt{\lambda_2} \\
&\qquad + b_3 \sin t\sqrt{\lambda_3} + b_3' \cos t\sqrt{\lambda_3} + \&c. \\
w &= c_1 \sin t\sqrt{\lambda_1} + \&c. \\
\&c. &= \&c.,
\end{aligned}
$$

where a_1, a_1', a_2, a_2', &c., are arbitrary, and b_1, b_1', b_2, b_2', &c., c_1, c_1', &c., are then determined by linear equations—the equations for determining b_1, c_1, &c., being the same as those for b_1', c_1', &c., except that the former contain a_1, and the latter a_1'.

If two roots λ_1 and λ_2 become equal,

$$
\begin{aligned}
u &= (a_1 + a_2) \sin t\sqrt{\lambda_1} + (a_1' + a_2') \cos t\sqrt{\lambda_1} + a_3 \sin t\sqrt{\lambda_3} \\
&\qquad + a_3' \cos t\sqrt{\lambda_3} + \&c. \\
v &= (b_1 + b_2) \sin t\sqrt{\lambda_1} + (b_1' + b_2') \cos t\sqrt{\lambda_1} + \&c. \\
\&c. &= \&c.;
\end{aligned}
$$

or, as the equations may be written,

$$
\left.
\begin{aligned}
u &= a_1 \sin t\sqrt{\lambda_1} + a_1' \cos t\sqrt{\lambda_1} + a_3 \sin t\sqrt{\lambda_3} \\
&\qquad + a_3' \cos t\sqrt{\lambda_3} + \&c. \\
v &= b_1 \sin t\sqrt{\lambda_1} + b_1' \cos t\sqrt{\lambda_1} + \&c. \\
w &= c_1 \sin t\sqrt{\lambda_1} + c_1' \cos t\sqrt{\lambda_1} + \&c.
\end{aligned}
\right\};\qquad (7)
$$

where a_1 and b_1 are arbitrary, but the remaining $(i-2)$ constants c_1, d_1, &c., are determined in terms of a_1 and b_1; also a_1' and b_1', are arbitrary, but c_1', d_1', &c., are connected with them by the same linear equations which give c_1, d_1, &c., in terms of a_1 and b_1.

The solution therefore is still complete, as it contains four arbitrary constants corresponding to the two equal roots of the equation $\Delta = 0$, in addition to the $2(i-2)$ arbitrary constants which arise from the $(i-2)$ remaining roots supposed unequal.

The preceding investigation can be easily extended when the equation $\Delta = 0$ has r roots equal to one another.

In this case $2r$ constants a_1, b_1, c_1, &c., a_1', b_1', c_1', &c., are arbitrary, and the i linear equations, which in general determine $(i-1)$ constants, in terms of the remaining one, fulfil the conditions required in order that only $i-r$ of them should be independent.

In fact, from what has been proved above it appears that, if D be the first minor of the determinant Δ, every double root of the equation $\Delta = 0$ must be a root of $D = 0$. Hence if Δ have r equal roots D must have $(r-1)$. Again, it is plain that D is related to its first minor in the same way in which Δ is related to D, and so on. We may therefore conclude that if the equation $\Delta = 0$ have r roots equal to λ_1 $(r-1)$ successive minors of Δ_1 must vanish.

The mode of investigation employed in the present Article is indicated by Thomson and Tait (*Natural Philosophy*, § 343 *m*).

For another method of arriving at similar results the reader is referred to Salmon's *Higher Algebra*, Art. 46, or to Routh's *Treatise on the Dynamics of a System of Rigid Bodies*, Part II., Art. 61.

264. **Directions of Harmonic Vibration.**—If we introduce the values of u, v, &c. from Article 259 into those of x, y, &c., in Article 258, we have

$$x = x_0 + k_1(a_x a_1 + b_x b_1 + c_x c_1 + \&c.)\sin(t\sqrt{\lambda_1} + \chi_1)$$
$$+ k_2(a_x a_2 + b_x b_2 + c_x c_2 + \&c.)\sin(t\sqrt{\lambda_2} + \chi_2) + \&c.,$$

$$y = y_0 + k_1(a_y a_1 + b_y b_1 + c_y c_1 + \&c.)\sin(t\sqrt{\lambda_1} + \chi_1)$$
$$+ k_2(a_y a_2 + b_y b_2 + c_y c_2 + \&c.)\sin(t\sqrt{\lambda_2} + \chi_2) + \&c.,$$

$$z = z_0 + k_1(a_z a_1 + b_z b_1 + c_z c_1 + \&c.)\sin(t\sqrt{\lambda_1} + \chi_1)$$
$$+ k_2(a_z a_2 + b_z b_2 + c_z c_2 + \&c.)\sin(t\sqrt{\lambda_2} + \chi_2) + \&c.$$

$$\&c. \quad = \quad \&c.$$

If we denote by $k_1\xi_1$, $k_2\xi_2$, &c., the increments of x corresponding to λ_1, λ_2, &c.; by $k_1\eta_1$, $k_2\eta_2$, &c., those of y, and so on, we may write these equations

$$\left.\begin{array}{l} x = x_0 + k_1\xi_1 + k_2\xi_2 + k_3\xi_3 + \&c. + k_i\xi_i \\ y = y_0 + k_1\eta_1 + k_2\eta_2 + k_3\eta_3 + \&c. + k_i\eta_i \\ z = z_0 + k_1\zeta_1 + k_2\zeta_2 + k_3\zeta_3 + \&c. + k_i\zeta_i \\ \qquad\qquad \&c. \quad = \quad \&c. \end{array}\right\}. \qquad (8)$$

Hence the motion of each particle is, in general, resolvable into i simple vibrations whose periods are

$$\frac{2\pi}{\sqrt{\lambda_1}}, \quad \frac{2\pi}{\sqrt{\lambda_2}}, \ldots \frac{2\pi}{\sqrt{\lambda_i}}.$$

The motion of any particle being determined, the motion of any other consists of simple vibrations having the same periods, or *harmonic* with the former.

The increments $k_1\xi_1$, $k_1\eta_1$, $k_1\zeta_1$, if they existed alone, would determine a direction of vibration for the particle x, y, z, which depends on the connexions between the particles and the forces acting on the system, but is independent of the initial positions and velocities. Hence the directions of the several component vibrations, as well as the ratios of their amplitudes for the different particles in any one harmonic set, depend on the particulars of the system (connexions and forces), and are independent of the particulars of the motion (initial positions and velocities).

The several systems of directions (ξ_1, η_1, ζ_1; ξ_1', η_1', ζ_1'; ξ_1'', η_1'', ζ_1''; &c.), (ξ_2, η_2, ζ_2; ξ_2', η_2', ζ_2'; &c.), &c., along which, if the particles were simultaneously displaced, they would all vibrate in the same period or harmonically, are termed *the directions of harmonic vibration.*

EXAMPLES.

In each of the following Examples up to 15, *with the exception of* 8, *the small oscillations of the given system are to be determined :—*

1. A material particle oscillates about the lowest point of a vertical circle under the action of gravity.

Take the lowest point for origin; let the vertical ordinate be z, and the horizontal x; then $2rz = x^2 + z^2$; and x being small, z is of the order x^2, and, accordingly, z^2 may be neglected; therefore

$$r\delta z = x\delta x, \quad r\frac{d^2z}{dt^2} = x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2;$$

hence $\frac{d^2z}{dt^2}$ is negligible, and

$$\left(m\frac{d^2x}{dt^2} - X\right)\delta x + \left(m\frac{d^2y}{dt^2} - Y\right)\delta y + \left(m\frac{d^2z}{dt^2} - Z\right)\delta z = m\frac{d^2x}{dt^2}\delta x + mg\,\delta z$$

$$= m\left(\frac{d^2x}{dt^2} + \frac{g}{r}x\right)\delta x;$$

therefore $$\frac{d^2x}{dt^2} + \frac{g}{r}x = 0, \text{ and } x = k\sin\left(\sqrt{\frac{g}{r}}\,t + \chi\right).$$

2. A heavy particle oscillates about the lowest point of a curve, lying in a vertical plane.

If r be the radius of curvature of the curve at its lowest point, the question is reduced to the last.

3. A heavy particle oscillates about the lowest point of a sphere.

Taking the origin at the lowest point, and the axis of z vertical, we have $2rz = x^2 + y^2 + z^2$. Accordingly we may neglect, as in Ex. 1, z^2 and $\frac{d^2z}{dt^2}$; also $r\delta z = x\delta x + y\delta y$;

hence $$\left(\frac{d^2x}{dt^2} + \frac{g}{r}x\right)\delta x + \left(\frac{d^2y}{dt^2} + \frac{g}{r}y\right)\delta y = 0;$$

therefore $$\frac{d^2x}{dt^2} + \frac{g}{r}x = 0, \text{ and } \frac{d^2y}{dt^2} + \frac{g}{r}y = 0;$$

therefore $$x = m\sin\left(\sqrt{\frac{g}{r}}\,t + \mu\right), \quad y = n\sin\left(\sqrt{\frac{g}{r}}\,t + \nu\right),$$

where m, n, μ, and ν are arbitrary constants. From these equations it appears that the projection of the path of the particle on a horizontal plane is an ellipse whose equation is

$$\left(\frac{x}{m}\right)^2 + \left(\frac{y}{n}\right)^2 - 2\frac{xy}{mn}\cos(\mu - \nu) = \sin^2(\mu - \nu).$$

The harmonic determinant is a quadratic function of λ having equal roots.

4. A heavy particle oscillates about the lowest point of an ellipsoid.

Take tangents to the lines of curvature at the lowest point for the axes of x and y, then the equation of the surface, neglecting small quantities of an order higher than x^2, is $2z = \frac{x^2}{R_1} + \frac{y^2}{R_2}$, where R_1 and R_2 are the principal radii of curvature. Substituting for δz in the general dynamical equation, neglecting $\frac{d^2z}{dt^2}$, and equating to zero the coefficients of δx and δy, we obtain

$$\frac{d^2x}{dt^2} + \frac{g}{R_1}x = 0, \quad \frac{d^2y}{dt^2} + \frac{g}{R_2}y = 0\,;$$

therefore $$x = m \sin\left(\sqrt{\frac{g}{R_1}}\,t + \mu\right), \quad y = n \sin\left(\sqrt{\frac{g}{R_2}}\,t + \nu\right).$$

A similar result holds good in the case of any concave surface.

5. A heavy bar hanging freely from one extremity is slightly displaced.

Take two horizontal lines at right angles to each other passing through the point of suspension for axes of x and y. Let the small angular displacements of the bar at any time round these axes be θ and ϕ; then, r being the distance of any point of the bar from its extremity,

$$x = r\phi, \quad y = r\theta, \quad z^2 = r^2 - x^2 - y^2\,;$$

therefore $$z = r\left(1 - \tfrac{1}{2}\theta^2 - \tfrac{1}{2}\phi^2\right), \quad \delta z = -r\left(\theta\,\delta\theta + \phi\,\delta\phi\right),$$

thus we may neglect $\frac{d^2z}{dt^2}$, and have

$$\Sigma mr^2\left(\frac{d^2\theta}{dt^2}\,\delta\theta + \frac{d^2\phi}{dt^2}\,\delta\phi\right) + g\Sigma mr\left(\theta\,\delta\theta + \phi\,\delta\phi\right) = 0.$$

Hence, $$\frac{d^2\theta}{dt^2}\int r^2\,dm + g\theta\int r\,dm = 0,$$

$$\frac{d^2\phi}{dt^2}\int r^2\,dm + g\phi\int r\,dm = 0.$$

6. Two balls connected by a horizontal bar, whose mass may be neglected, are suspended by two vertical cords of equal length. The bar receives a slight displacement of rotation round a vertical axis midway between the cords.

Let ψ be the angle which the bar makes with a horizontal line parallel to its initial position, θ the inclination of one of the cords to the vertical (see figure in Ex. 7, Art. 228), l its length, b the distance from the middle point of the bar to one of the balls; then

$$x = b\cos\psi = b\left(1 - \tfrac{1}{2}\psi^2\right), \quad y = b\psi, \quad z = l\left(1 - \tfrac{1}{2}\theta^2\right)\,;$$

but $$l\theta = b\psi; \quad \therefore\ z = l\left(1 - \tfrac{1}{2}\frac{b^2}{l^2}\psi^2\right),$$

and $$x' = -x, \quad y' = -y, \quad z' = z;$$

then $$\frac{d^2x}{dt^2}, \quad \frac{d^2x'}{dt^2}, \quad \frac{d^2z}{dt^2}, \quad \frac{d^2z'}{dt^2}$$

may be neglected, and equating to cipher the coefficient of $\delta\psi$ in the general equation, we have

$$(m + m')\,b^2\frac{d^2\psi}{dt^2} + g\,(m + m')\frac{b^2}{l}\,\psi = 0;$$

therefore $$\psi = \alpha \sin\left(\sqrt{\frac{g}{l}}\,t + \chi\right),$$

where α and χ are arbitrary constants.

This shows that the period of vibration is the same as that of a pendulum whose length is l.

7. A heavy bar having no balls attached is suspended and displaced as in the preceding example.

Let r be the distance of any point of the bar from its centre, and b the distance from its centre to the point of attachment of one of the cords; then, as in the preceding example,

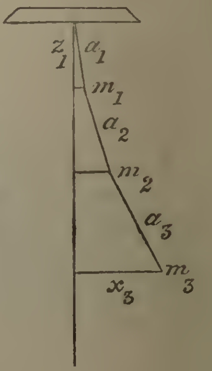
$$\frac{d^2\psi}{dt^2}\int r^2\,dm + g\,\frac{b^2}{l}\,\psi\int dm = 0;$$

therefore $$\psi = \alpha \sin\left(\frac{b}{k}\sqrt{\frac{g}{l}}\,t + \chi\right), \text{ where } \int r^2\,dm = mk^2.$$

8. How must the bar in the preceding example be suspended in order that its vibrations should be isochronous with those of a ball hung by one of the supporting cords?

Ans. $b = k$. In the case of a homogeneous bar whose length is $2a$, $b = \dfrac{a}{\sqrt{3}}$.

9. A number of balls suspended by a fine cord hang in a vertical line, and are slightly displaced in the same vertical plane.



Let x_1, z_1; x_2, z_2, &c., be the horizontal and vertical coordinates of the balls; a_1, a_2, &c., the distances from the point of suspension to the first ball, from the first ball to the second, and so on; θ_1, θ_2, &c., the angles which a_1, a_2, &c., make with the vertical at any instant. The weight of the cord being neglected, the distances a_1, &c. are invariable; then

$$x_1 = a_1\theta_1, \qquad z_1 = a_1\cos\theta_1 = a_1(1 - \tfrac{1}{2}\theta_1^2),$$

$$x_2 = a_1\theta_1 + a_2\theta_2, \quad z_2 = a_1(1 - \tfrac{1}{2}\theta_1^2) + a_2(1 - \tfrac{1}{2}\theta_2^2), \ \&c.$$

Substituting in the general dynamical equation, neglecting $\frac{d^2z_1}{dt^2}$, &c., and equating to zero the coefficients of $\delta\theta_1$, $\delta\theta_2$, &c., we have, after dividing the first equation by a_1, the second by a_2, &c.,

$$(m_1+m_2+m_3+\&c.)\,a_1\ddot{\theta}_1+(m_2+m_3+\&c.)\,a_2\ddot{\theta}_2+(m_3+\&c.)\,a_3\ddot{\theta}_3+\&c.$$
$$+(m_1+m_2+\&c.)\,g\theta_1=0,$$

$$(m_2+m_3+\&c.)\,a_1\ddot{\theta}_1+(m_2+m_3+\&c.)\,a_2\ddot{\theta}_2+(m_3+\&c.)\,a_3\ddot{\theta}_3+\&c.$$
$$+(m_2+\&c.)\,g\theta_2=0,$$

. . . .

$$m_n a_1\ddot{\theta}_1+m_n a_2\ddot{\theta}_2+m_n a_3\ddot{\theta}_3+\&c.+m_n a_n\ddot{\theta}_n+m_n g\theta_n=0.$$

Hence, assuming

$$\theta_1=k\alpha\sin(t\sqrt{\lambda}+\chi),\quad \theta_2=k\beta\sin(t\sqrt{\lambda}+\chi),\ \&c.,$$

where k and χ are arbitrary constants, we get to determine α, β, γ, &c., and λ the equations

$$(m_1+m_2+\&c.)(a_1\lambda-g)\alpha+(m_2+m_3+\&c.)\,a_2\lambda\beta+(m_3+\&c.)\,a_3\lambda\gamma+\&c.=0,$$

$$(m_2+m_3+\&c.)\,a_1\lambda\alpha+(m_2+m_3+\&c.)(a_2\lambda-g)\beta+(m_3+\&c.)\,a_3\lambda\gamma+\&c.=0,$$

. . . .

$$(a_1\alpha+a_2\beta+a_3\gamma+\&c.+a_n\omega)\lambda-g\omega=0.$$

10. The system of balls suspended as in the last example are displaced in different vertical planes.

In this case, θ_1 and ϕ_1 being the angular displacements of a_1 round the axes of x and y, θ_2 and ϕ_2 those of a_2, &c.,

$$x_1=a_1\phi_1,\quad y_1=a_1\theta_1,\quad z_1=a_1(1-\tfrac{1}{2}\theta_1^2-\tfrac{1}{2}\phi_1^2),\ \&c.;$$

then θ_1, θ_2, &c., are independent of ϕ_1, ϕ_2, &c., as in Ex. 3. The values of θ_1, &c., are therefore the same as in the last example, whilst those of ϕ_1, &c., differ from them only by having a different set of arbitrary constants.

11. A number of rigid bars are hinged together in the same vertical line, and are displaced in the same vertical plane.

If x_1, z_1 be the coordinates of *any* point of the first bar, x_2, z_2 those of any point of the second, &c., and θ_1, θ_2, &c., the inclinations of the bars to the vertical; $x_1=r_1\theta_1$, $z_1=r_1(1-\frac{1}{2}\theta_1^2)$, $x_2=a_1\theta_1+r_2\theta_2$, $z_2=a_1(1-\frac{1}{2}\theta_1^2)+r_2(1-\frac{1}{2}\theta_2^2)$, &c.

Proceeding as in Ex. 9, we obtain

$$\{m_1k_1^2+(m_2+m_3+\&c.)a_1^2\}\ddot{\theta}_1+\{m_2b_2+(m_3+\&c.)a_2\}a_1\ddot{\theta}_2$$

$$+\{m_3b_3+(m_4+\&c.)a_3\}a_1\ddot{\theta}_3+\&c.+\{m_1b_1+(m_2+m_3+\&c.)a_1\}g\theta_1=0,$$

$$\{m_2b_2+(m_3+\&c.)a_2\}a_1\ddot{\theta}_1+\{m_2k_2^2+(m_3+\&c.)a_2^2\}\ddot{\theta}_2$$

$$+\{m_3b_3+(m_4+\&c.)a_3\}a_2\ddot{\theta}_3+\&c.+\{m_2b_2+(m_3+m_4+\&c.)a_2\}g\theta_2=0,$$

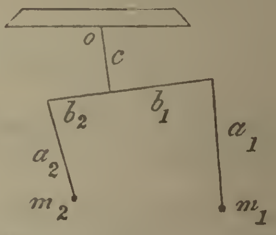
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$m_nb_na_1\ddot{\theta}_1+m_nb_na_2\ddot{\theta}_2+m_nb_na_3\ddot{\theta}_3+\&c.+m_nk_n^2\ddot{\theta}_n+m_nb_ng\theta_n=0,$$

where k_1, k_2, &c., are the radii of gyration of the bars round their extremities, and b_1, b_2, &c., the distances from the extremities of their centres of inertia. The solution of these equations is obtained in the same manner as in Ex. 9.

12. Two balls, m_1, and m_2, are suspended by cords, whose lengths are a_1 and a_2, from the extremities of the arms b_1 and b_2 of a lever, whose position of equilibrium is horizontal, and which hangs by a bar c, at right angles to it, from a fixed point. The system is displaced in the vertical plane which it occupies when in equilibrium.

Let θ_1, θ_2, θ_3, be the inclinations of a_1, a_2, and c to the vertical; x_1, z_1, and x_2, z_2 the coordinates of m_1 and m_2; and x_3, z_3 those of *any* point of the lever whose distance from the point of junction with c is r; then



$$x_1=a_1\theta_1+b_1(1-\tfrac{1}{2}\theta_3^2)+c\theta_3,\quad z_1=a_1(1-\tfrac{1}{2}\theta_1^2)-b_1\theta_3+c(1-\tfrac{1}{2}\theta_3^2),$$

$$x_2=a_2\theta_2-b_2(1-\tfrac{1}{2}\theta_3^2)+c\theta_3,\quad z_2=a_2(1-\tfrac{1}{2}\theta_2^2)+b_2\theta_3+c(1-\tfrac{1}{2}\theta_3^2),$$

$$x_3=r(1-\tfrac{1}{2}\theta_3^2)+c\theta_3,\quad z_3=-r\theta_3+c(1-\tfrac{1}{2}\theta_3^2).$$

Expressing $\frac{d^2x_1}{dt^2}$, δx_1, &c., in terms of θ_1, θ_2, θ_3, the general equation of dynamics becomes

$$m_1(a_1\ddot{\theta}_1+c\ddot{\theta}_3)(a_1\delta\theta_1+c\delta\theta_3)+m_2(a_2\ddot{\theta}_2+c\ddot{\theta}_3)(a_2\delta\theta_2+c\delta\theta_3)$$

$$+(m_3c^2+m_1b_1^2+m_2b_2^2+\int r^2\,dm_3)\ddot{\theta}_3\,\delta\theta_3$$

$$+g\{m_1a_1\theta_1\delta\theta_1+m_2a_2\theta_2\delta\theta_2+[(m_1+m_2+m_3)c\theta_3+m_1b_1-m_2b_2+\int r\,dm_3]\,\delta\theta_3\}$$

$=0,$ where m_3 is the mass of the lever.

As the initial position is one of equilibrium,

$$m_1b_1-m_2b_2+\int r\,dm_3=0.$$

Hence, putting

$$m_1 b_1^2 + m_2 b_2^2 + \int r^2\, dm_3 + (m_1 + m_2 + m_3) c^2 = (m_1 + m_2 + m_3) c a_3,$$

we have, from the equations got by equating the coefficients of $\delta\theta_1$, &c. to cipher,

$$\theta_1 = k\alpha \sin(t\sqrt{\lambda} + \chi), \quad \theta_2 = k\beta \sin(t\sqrt{\lambda} + \chi), \quad \theta_3 = k\gamma \sin(t\sqrt{\lambda} + \chi),$$

where α, β, γ, and λ are determined by the equations

$$(a_1\lambda - g)\,\alpha + c\lambda\gamma = 0,$$

$$(a_2\lambda - g)\,\beta + c\lambda\gamma = 0,$$

$$m_1 a_1 \lambda\alpha + m_2 a_2 \lambda\beta + (m_1 + m_2 + m_3)\,(a_3\lambda - g)\,\gamma = 0.$$

The resulting cubic for λ becomes a quadratic if either $a_1 = 0$, or $a_2 = 0$; and appears as the product of a linear and a quadratic factor if $a_1 = a_2$.

13. The balls and cords in the preceding example are replaced by two bars which hang freely from the ends of the lever.

x_1, z_1 being the coordinates of *any* point of the first bar whose distance from the extremity is r_1; x_2, z_2, r_2, corresponding quantities for the second bar, and the rest of the notation as in the last example, we have

$$x_1 = r_1\theta_1 + b_1(1 - \tfrac{1}{2}\theta_3^2) + c\theta_3, \quad z_1 = r_1(1 - \tfrac{1}{2}\theta_1^2) - b_1\theta_3 + c(1 - \tfrac{1}{2}\theta_3^2),$$

$$x_2 = r_2\theta_2 - b_2(1 - \tfrac{1}{2}\theta_3^2) + c\theta_3, \quad z_2 = r_2(1 - \tfrac{1}{2}\theta_2^2) + b_2\theta_3 + c(1 - \tfrac{1}{2}\theta_3^2),$$

and x_3 and z_3 the same as in the preceding example. Hence, assuming as before

$$\theta_1 = k\alpha \sin(t\sqrt{\lambda} + \chi), \text{ &c.},$$

and putting

$$\int r_1\, dm_1 = l_1, \quad \int r_2\, dm_2 = l_2, \quad \frac{\int r_1^2 dm_1}{\int r_1\, dm_1} = a_1, \quad \frac{\int r_2^2\, dm_2}{\int r_2\, dm_3} = a_2,$$

we have, to determine α, β, γ, and λ, the equations

$$(a_1\lambda - g)\,\alpha + c\lambda\gamma = 0,$$

$$(a_2\lambda - g)\beta + c\lambda\gamma = 0,$$

$$m_1 l_1 \lambda\alpha + m_2 l_2 \lambda\beta + (m_1 + m_2 + m_3)\;(a_3\lambda - g) = 0.$$

When $a_1 = a_2$, the cubic for λ appears as the product of a linear and a quadratic factor.

14. A rigid body, having a fixed point and in stable equilibrium under the action of a conservative system of forces, is slightly disturbed.

Let the axis, a rotation round which would bring the body from its position of equilibrium to its actual position at any time (Art. 229), make angles with the principal axes of the body at the fixed point whose direction cosines are l, m, n. Let σ be the magnitude of the required rotation, which by hypothesis is a small quantity. The coordinates of each point of the body are then at any time functions of constants and of the variables, σ, l, m, n, or, of the three independent variables σl, σm, σn, which may be denoted by θ, ϕ, ψ.

Hence, as θ, ϕ, ψ are small, and the initial position is one of equilibrium (Art. 258),

$$V = V_0 - \tfrac{1}{2}(p_{11}\theta^2 + p_{22}\phi^2 + p_{33}\psi^2 + 2p_{12}\theta\phi + 2p_{13}\theta\psi + 2p_{23}\phi\psi).$$

Again, neglecting small quantities of the second order, $\omega_1 = \dot\theta$, $\omega_2 = \dot\phi$, $\omega_3 = \dot\psi$; and therefore (Art. 245)

$$T = \tfrac{1}{2}(A\dot\theta^2 + B\dot\phi^2 + C\dot\psi^2),$$

neglecting small quantities of the third order.

Hence Lagrange's equations (Art. 260) become

$$A\ddot\theta + p_{11}\theta + p_{12}\phi + p_{13}\psi = 0,$$

$$B\ddot\phi + p_{12}\theta + p_{22}\phi + p_{23}\psi = 0,$$

$$C\ddot\psi + p_{13}\theta + p_{23}\phi + p_{33}\psi = 0.$$

Assuming

$$\theta = k\alpha \sin(t\sqrt{\lambda} + \chi), \quad \phi = k\beta \sin(t\sqrt{\lambda} + \chi), \quad \psi = k\gamma \sin(t\sqrt{\lambda} + \chi),$$

we have, for the determination of α, β, γ, λ, the equations

$$A\lambda\alpha = p_{11}\alpha + p_{12}\beta + p_{13}\gamma,$$

$$B\lambda\beta = p_{12}\alpha + p_{22}\beta + p_{23}\gamma,$$

$$C\lambda\gamma = p_{13}\alpha + p_{23}\beta + p_{33}\gamma.$$

If α_1, α_2, &c., be the values of α, &c., corresponding to λ_1 and λ_2, two of the roots of the cubic for λ, it is easy to see that

$$(\lambda_1 - \lambda_2)(A\alpha_1\alpha_2 + B\beta_1\beta_2 + C\gamma_1\gamma_2) = 0;$$

hence $$A\alpha_1\alpha_2 + B\beta_1\beta_2 + C\gamma_1\gamma_2 = 0,$$

and therefore also

$$\alpha_2(p_{11}\alpha_1 + p_{12}\beta_1 + p_{13}\gamma_1) + \beta_2(p_{12}\alpha_1 + p_{22}\beta_1 + p_{23}\gamma_1) + \gamma_2(p_{13}\alpha_1 + p_{23}\beta_1 + p_{33}\gamma_1) = 0.$$

Accordingly the lines whose direction cosines are proportional to α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3; are conjugate diameters of the momental ellipsoid, and likewise of the quadric E, whose equation referred to the principal axes of the body at the fixed point is

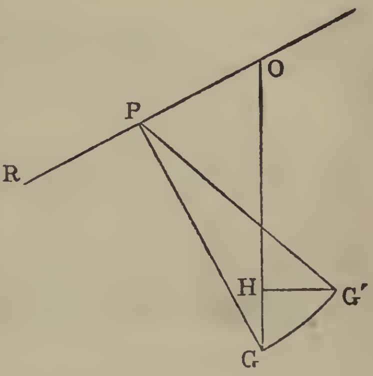
$$p_{11}x^2 + p_{22}y^2 + p_{33}z^2 + 2p_{12}xy + 2p_{13}xz + 2p_{23}yz = K.$$

Since the initial position is one of stable equilibrium, E must be an ellipsoid (Art. 262).

An angular displacement σ_1 from the position of equilibrium brings the body into a position whose potential energy, relative to the initial position, is $\frac{1}{2}K\left(\frac{\sigma_1}{r_1}\right)^2$, r_1 being the semi-diameter of E round which the rotation σ_1 is effected. Hence all small angular displacements, which are proportional to the diameters of E round which they are effected, bring the body into positions having the same potential energy. On this account Dr. Ball calls E the *ellipsoid of equal energy*, or, the *potential ellipsoid* corresponding to the initial position of equilibrium. The results arrived at may therefore be stated as follows:—The *harmonic axes of a rigid body, having a fixed point and in stable equilibrium under the action of a conservative system of forces, are the three conjugate diameters common to the momental and the potential ellipsoids.* This theorem is due to Dr. Ball.

15. If gravity be the only force acting on the body in the last example, show that the potential ellipsoid becomes a circular cylinder, and determine the positions of the harmonic axes.

Let O be the fixed point, G the initial position of the centre of inertia, G' its position resulting from a small angular displacement σ of the body round the line OR whose direction cosines are l, m, n. Draw GP and $G'P$ perpendicular to OR, and $G'H$ perpendicular to OG. Then $\sigma = \angle GPG'$, and the potential energy due to the displacement is $\mathfrak{M}g \,.\, GH$. Putting $OG = a$, and $\angle GOP = \rho$, we have



$$GH = \frac{GG'^2}{2a} = \frac{PG^2 \,.\, \sigma^2}{2a} = \frac{a}{2}\sigma^2 \sin^2\rho.$$

Again, if λ, μ, ν be the direction cosines of OG,

$$\sigma \cos\rho = \sigma(l\lambda + m\mu + n\nu) = \theta\lambda + \phi\mu + \psi\nu.$$

Hence, the potential energy due to the displacement is

$$\tfrac{1}{2}\mathfrak{M}ga\,\{\theta^2 + \phi^2 + \psi^2 - (\theta\lambda + \phi\mu + \psi\nu)^2\},$$

and the quadric E is determined by the equation

$$x^2 + y^2 + z^2 - (\lambda x + \mu y + \nu z)^2 = K,$$

which represents a right circular cylinder having OG for its axis.

This can also be easily seen directly as follows:—The amount of energy T required to turn the body through an angle σ round any semidiameter r of E is, as we have seen, $\frac{1}{2}K\frac{\sigma^2}{r^2}$. Now, if r be vertical, $T = 0$, and therefore $r = \infty$. Also, if r be horizontal, $T = \mathfrak{M}gh$ (where h is the height through which the centre of inertia is raised), and is therefore constant if σ be constant, therefore r is constant; accordingly the corresponding section of E is a circle.

To determine the harmonic axes—One is the vertical, OG, the other two are found as follows:—Draw the diametral plane of the momental ellipsoid which is conjugate to a vertical line through the fixed point; it will meet the cylinder E and the momental ellipsoid in two ellipses; the pair of conjugate diameters common to these two are the lines required. Also, since a horizontal section of E is a circle, the projections on any horizontal plane of the two non-vertical harmonic axes are lines at right angles to each other. If the body be displaced without initial velocity, there will be no oscillation round the vertical axis; and if the periods of vibration round the other two axes be different, the instantaneous axis of rotation will either oscillate or revolve continuously in the plane of the non-vertical harmonic axes. If after displacement an initial velocity be imparted to the body, in order that there should be small oscillations, this initial velocity of rotation must be round an axis in the plane of the two non-vertical harmonic axes.

16. If the system of forces acting on the body be constant in magnitude and direction, determine the values of p_{11}, &c., in Example 14.

Let the components of the constant force acting at any point of the body, parallel to the initial directions of its principal axes, be X, Y, Z; and let the coordinates of the point of application referred to the principal axes be x, y, z, then,

$$p_{11} = \Sigma(yY + zZ), \quad p_{22} = \Sigma(zZ + xX), \quad p_{33} = \Sigma(xX + yY),$$

$$2p_{12} = -\Sigma(xY + yX), \quad 2p_{13} = -\Sigma(xZ + zX), \quad 2p_{23} = -\Sigma(yZ + zY).$$

17. If a system whose position is determined by three independent variables perform small oscillations, prove that the harmonic axes are the three conjugate diameters common to the quadrics $\mathfrak{T}$ and S (Art. 261).

CHAPTER XII.

THE DYNAMICAL PRINCIPLES.

Section I.—*Motion of Centre of Inertia, and Moments of Momentum.*

265. **Centre of Inertia of a Free System.**—A system is said to be free when it is not acted on by any restraints external to itself.

It has been shown in Art. 207 that the equations of motion of a rigid body are in their most general form applicable to any system whatever. Hence, from Art. 204, we may conclude that—

In any free system the centre of inertia moves as if all the forces were applied to the entire mass concentrated at that point.

As this theorem is one of great importance, we shall prove it directly from D'Alembert's Principle as follows:—

If a system be free, equal and parallel displacements in any arbitrary direction can be given to each of its different points, as their relative positions are thereby undisturbed. In D'Alembert's equation (Art. 201) we may therefore consider δx, δy, δz, as arbitrary and independent, but as having the same values for all points of the system. Hence

$$\Sigma\left(X - m\frac{d^2x}{dt^2}\right) = 0, \quad \Sigma\left(Y - m\frac{d^2y}{dt^2}\right) = 0, \quad \Sigma\left(Z - m\frac{d^2z}{dt^2}\right) = 0,$$

and therefore

$$(\Sigma m)\frac{d^2\bar{x}}{dt^2} = \Sigma X, \; (\Sigma m)\frac{d^2\bar{y}}{dt^2} = \Sigma Y, \; (\Sigma m)\frac{d^2\bar{z}}{dt^2} = \Sigma Z. \qquad (1)$$

266. **Restraints and Partial Freedom.**—If a system be subject to restraints external to itself, we may apply equations (1), provided we suppose the restraints replaced by the forces to which they give rise.

If a system, though not entirely free, be such that equal and parallel displacements of arbitrary magnitude can be given to each of its points in a definite direction, and if the axis of x be taken in that direction, we have still the equation

$$(\Sigma m)\frac{d^2\bar{x}}{dt^2} = \Sigma X.$$

267. **Internal Forces.**—Any force by which two parts of a system act on each other is said to be internal.

Since action and reaction are equal and opposite, the components of internal forces destroy one another in the sums ΣX, ΣY, and ΣZ. Hence in any system—

Internal forces have no effect on the motion of the centre of inertia.

268. **Case of no External Forces.**—It follows from Art. 267, that if a system be acted on by no external forces, its centre of inertia is either at rest or moves in a straight line with a constant velocity. This theorem is sometimes termed *The Principle of the Conservation of the Motion of the Centre of Inertia.*

Results similar to those of the preceding Articles hold good for impulses.

269. **Motion of a Free System relative to its Centre of Inertia.**—If ξ, η, ζ be the coordinates of any point of a system referred to axes through its centre of inertia parallel to fixed directions, $x = \bar{x} + \xi$, $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$. Substituting in D'Alembert's equation, we have

$$\delta\bar{x}\left(\Sigma X - (\Sigma m)\frac{d^2\bar{x}}{dt^2}\right) + \delta\bar{y}\left(\Sigma Y - (\Sigma m)\frac{d^2\bar{y}}{dt^2}\right) + \delta\bar{z}\left(\Sigma Z - (\Sigma m)\frac{d^2\bar{z}}{dt^2}\right)$$

$$- \delta\bar{x}\,\Sigma m\frac{d^2\xi}{dt^2} - \delta\bar{y}\,\Sigma m\frac{d^2\eta}{dt^2} - \delta\bar{z}\,\Sigma m\frac{d^2\zeta}{dt^2}$$

$$- \frac{d^2\bar{x}}{dt^2}\,\Sigma m\delta\xi - \frac{d^2\bar{y}}{dt^2}\,\Sigma m\delta\eta - \frac{d^2\bar{z}}{dt^2}\,\Sigma m\,\delta\zeta$$

$$+ \Sigma\left\{\left(X - m\frac{d^2\xi}{dt^2}\right)\delta\xi + \left(Y - m\frac{d^2\eta}{dt^2}\right)\delta\eta + \left(Z - m\frac{d^2\zeta}{dt^2}\right)\delta\zeta\right\} = 0.$$

Now $\Sigma m\xi = \Sigma m\eta = \Sigma m\zeta = 0$: hence, by equations (1), Art. 265, we obtain

$$\Sigma\left\{\left(X - m\frac{d^2\xi}{dt^2}\right)\delta\xi + \left(Y - m\frac{d^2\eta}{dt^2}\right)\delta\eta + \left(Z - m\frac{d^2\zeta}{dt^2}\right)\delta\zeta\right\} = 0. \quad (2)$$

It follows from this equation that—

The motion of a free system relative to its centre of inertia is the same as if this point were fixed in space, the applied forces being unaltered as regards magnitude, direction, and point of application.

270. **Moments of Momentum.**—If x, y, z be the coordinates of any point of a system, and m the mass concentrated at that point, the quantity

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

is called the *moment of momentum* of the system round the axis of z. It may also be termed twice the areal velocity of the system round the same axis (Arts. 29, 206).

The Momentum Axis and Principal Plane of any system are determined as in Art. 253. It can be shown, as has been done in that Article for a rigid body, that—

The moment of momentum of a system round an axis through any point O is equal to the moment of momentum round a parallel axis through the centre of inertia, together with the moment of momentum relative to O of the entire mass of the system supposed to be concentrated at the centre of inertia, and moving with it.

271. **Free System.**—Equations (8) obtained in Article 206, for determining the moments of momentum of a rigid body, may, like those for determining the motion of the centre of inertia, be applied to any free system. This appears from Art. 207.

The direct deduction of these equations from D'Alembert's Principle is effected, in the case of any system, by giving to each point of the system a displacement such as it would receive if all the points were rigidly connected, and the whole system made to rotate round an arbitrary axis. Whatever be the internal connexions of the system, such displacements are possible, since they leave the relative positions of all the points unaltered.

272. **Restraints and Partial Freedom.**—A system which is not free may be regarded as free if the external restraints be replaced by the forces to which they give rise.

If the system, though not free, be such that displacements can be given to its points, such as they would receive if they formed a rigid system made to rotate round a fixed axis, the equation for determining the moment of momentum of the system round this axis still holds good. Hence,

If there be a fixed line in a system, the rate of change, relative to the time, in the moment of momentum of the system round this line is equal to the moment of the applied forces.

If there be one point of a system fixed, Equations (8) Art. 206 hold good for this point as origin.

273. **Internal Forces.**—Since internal forces occur in pairs, each pair consisting of two equal and opposite forces having a common line of direction, the moment round any line of the whole set of internal forces must be zero. Hence *the moments of momentum of any system are unaffected by forces internal to the system.*

274. **Conservation of Moment of Momentum.**—If a free system be unacted on by any forces external to itself, its resultant moment of momentum, relative to any point fixed in space, is constant, and has for its axis a line whose direction is invariable.

A similar result holds good for the centre of inertia even though this point be not fixed in space.

If a system, otherwise free, contain a point or a line fixed in space, and be unacted on by external forces, the resultant moment of momentum of the system relative to the fixed point, or the moment of momentum round the fixed line, is constant.

The theorems enunciated in this Article together constitute what has been often termed *The Principle of the Conservation of the Moment of Momentum*, or *The Principle of the Conservation of Areas.*

As the moment of a force round an axis intersecting the line of direction of the force is zero, we see that—

If the lines of direction of all the external forces which act on a free system be met by the same space-axis, the moment of momentum of the system round this axis is constant.

If the space-axis be fixed in the system, which is otherwise free, the theorem above still holds good.

EXAMPLES.

1. A number of spheres are projected in different directions with different initial velocities along a rough horizontal plane; find the path of their common centre of inertia.

Ans. A series of parabolas, and finally a straight line (see (1), Ex. 13, p. 278).

2. Show that the centre of inertia of the universe is either fixed in space or else moves in a straight line with a constant velocity.

3. A dog walks from one end to the other of a uniform plank which is placed on a smooth horizontal table; determine the displacement of the plank.

Let a be the length of the plank, P its mass, D that of the dog; the displacement is $\frac{D}{D+P}a$.

4. A uniform plank is placed on a smooth inclined plane so as to be perpendicular to the intersection of the inclined plane with the horizon; determine the time in which a dog should go from the upper to the lower end of the plank in order that it should remain unmoved.

Let t be the time required. The displacement of the centre of inertia of the system in the time t in space is $\frac{1}{2}gt^2 \sin i$, and relative to the plank is $\frac{D}{D+P}a$. If the plank remain unmoved these must be equal. Hence

$$t^2 = \frac{2D}{D+P} \cdot \frac{a}{g \sin i}.$$

5. A sphere is projected with a velocity v along a uniform smooth tube within which it fits exactly. The tube rests on a smooth horizontal plane, and its axis forms a circle; determine the motion.

Let m be the mass of the sphere, m' that of the tube, and a the radius of the circle formed by its axis. The common centre of inertia O of the tube and sphere moves parallel to the direction of projection of the sphere with a velocity $\frac{mv}{m+m'}$, and the centres of the tube and sphere describe circles round O with an angular velocity $\frac{v}{a}$.

6. A uniform circular plate whose centre is fixed lies on a smooth horizontal plane. An insect starts from the centre of the plate, and returns to the same point after describing a circle whose diameter is the radius of the plate; find the angle through which the plate has turned.

Let ϕ be the angle through which the plate has turned at any time, a its radius, m its mass, m' that of the insect, r and θ its polar coordinates in space, r and ψ its polar coordinates relative to the plate; then

$$m\frac{a^2}{2}\frac{d\phi}{dt} + m'r^2\frac{d\theta}{dt} = 0, \quad \psi = \theta - \phi, \quad r = a\cos\psi.$$

Hence
$$\phi = -\int \frac{2m'\cos^2\psi}{m + 2m'\cos^2\psi}\,d\psi,$$

and the angle required is

$$\pi\left\{1 - \left(\frac{m}{m + 2m'}\right)^{\frac{1}{2}}\right\}.$$

7. A satellite of mass m is moving in a circle whose radius is r, round a planet whose mass is M, and which rotates round an axis at right angles to the plane of the orbit with an angular velocity n. If C be the moment of inertia of the planet, and μ the attraction between unit masses at the unit of distance, show that the moment of momentum of the system round its centre of inertia is

$$C\left\{n + \mu^{\frac{1}{2}}\frac{Mm}{C}(M + m)^{-\frac{1}{2}}r^{\frac{1}{2}}\right\}.$$

8. A hollow cylinder filled with water is projected without initial rotation in a direction perpendicular to its axis, along a rough horizontal plane; determine the time at which pure rolling begins, the amount of friction subsequently brought into play, and the time at which the cylinder comes to rest.

Let M be the mass of the cylinder and contained water, I the moment of inertia of the cylinder round its central axis, a the radius of its external surface, μ and f the coefficients of sliding and rolling friction, v the initial velocity of the common centre of inertia G of the cylinder and contained water, t_1 the time at which pure rolling begins, F the friction subsequently brought into play, v_1 the velocity of the point G at the time t_1, t_2, the time at which the cylinder comes to rest. Then we find

$$t_1 = \frac{1}{\mu + \lambda\left(\mu - \frac{f}{a}\right)}\frac{v}{g}, \quad F = \frac{\lambda}{1 + \lambda}\frac{f}{a}Mg, \quad v_1 = \frac{\lambda\left(\mu - \frac{f}{a}\right)}{\mu + \lambda\left(\mu - \frac{f}{a}\right)}v, \quad t_2 = \frac{a}{f}\frac{v}{g},$$

where $\lambda I = Ma^2$. In these equations λ must exceed 2, which is its value in the case of a solid cylinder. We see that as λ increases, t_1 diminishes, F and v_1 increase, whilst t_2 remains constant, being the same as in the case of a solid cylinder (see Ex. 13, p. 278).

9. A smooth rod having one extremity fixed moves on a smooth horizontal table, and drives a particle of mass equal to its own, which starts from rest from a point indefinitely near the fixed extremity of the rod. Find the inclination of the rod to the direction of motion of the particle when the latter has reached any definite point of the rod.

As the moment of momentum and vis viva of the system are constant (see Art. 209), we have, if ψ be the angle required, $\tan\psi = \dfrac{k}{\sqrt{(r^2+k^2)}}$, where k is the radius of gyration of the rod, and r the distance of the particle from the fixed extremity.

10. A triangular prism rests with one rectangular face on a smooth horizontal plane. A rough cylinder, having its axis parallel to the edge of the prism, rolls down one of its faces starting from rest, the centres of inertia of the prism and cylinder being in the same vertical plane; determine the angular velocity of the cylinder when it reaches the horizontal plane, and the distance through which the prism has moved.

Let the axis of x be the intersection of the horizontal plane with the vertical plane containing the centres of inertia, the axis of z being vertical. Let x' be the coordinate of the centre of inertia of the prism, m' its mass; x, z the coordinates of the centre of the cylinder, a its radius, m its mass, k its radius of gyration, ϕ the angle through which it has turned, and s the distance on the prism perpendicular to its edge through which the line of contact of the cylinder has moved, at any time; then, i being the angle which the face of the prism makes with the horizontal plane,

$$x - x' = x_0 - x_0' + s\cos i, \quad z = z_0 - s\sin i, \quad s = a\phi;$$

and from the conservation of the motion of the centre of inertia and the equation of vis viva, we have

$$mx + m'x' = mx_0 + m'x_0', \quad m'\dot{x}'^2 + m(\dot{x}^2 + \dot{z}^2 + k^2\dot{\phi}^2) = 2gm(z_0 - z).$$

Hence, if h be the initial height of the centre of the cylinder, and ω its angular velocity when it reaches the horizontal plane,

$$\omega^2 = \frac{2(m+m')g(h-a)}{(m' + m\sin^2 i)a^2 + (m+m')k^2}, \quad x' - x_0' = -\frac{m}{m+m'}(h-a)\cot i.$$

SECTION II.—*Energy.*

275. **Energy.**—Work and Energy have been defined, Arts. 116 & 126, and the equation of Energy for a rigid body has been obtained by two different methods (Arts. 129 & 208). In the present section we propose to consider the subject of Energy in a somewhat more general manner, and to show that on the equation of Energy may be based the whole theory of the action of forces on a connected system.

276. **Equation of Energy.**—If any system of material points be acted on by any forces, we may suppose the constraints and connexions of the system replaced by corresponding forces, and with this proviso regard each point as entirely free. Assuming then the principles which govern the resolution and composition of forces acting at a point, and the relations between force and acceleration, we have

$$m_1 \frac{d^2x_1}{dt^2} = X_1, \quad m_1 \frac{d^2y_1}{dt^2} = Y_1, \quad m_1 \frac{d^2z_1}{dt^2} = Z_1,$$

$$m_2 \frac{d^2x_2}{dt^2} = X_2, \quad m_2 \frac{d^2y_2}{dt^2} = Y_2, \quad m_2 \frac{d^2z_2}{dt^2} = Z_2,$$

$$\&c., \qquad\qquad \&c.,$$

where X_1, Y_1, Z_1, include the components of the forces which replace the constraints, if any, acting at x_1, y_1, z_1, and so for the other points of the system.

Multiplying the first equation by dx_1, the second by dy_1, &c., and adding, we have

$$\Sigma m \left(\frac{d^2x}{dt^2} dx + \frac{d^2y}{dt^2} dy + \frac{d^2z}{dt^2} dz \right) = \Sigma (X dx + Y dy + Z dz);$$

or, putting $T = \frac{1}{2} \Sigma m v^2$,

$$\frac{dT}{dt} . dt = \Sigma (X dx + Y dy + Z dz). \tag{1}$$

In virtue of this equation, T is called the kinetic energy of the system (Art. 127).

277. **Conservation of Energy.**—*If the mutual forces between the parts of a material system are independent of their velocities, the system must be conservative* (Art. 122).

To prove this we have to show that in going from the configuration A to the configuration B, the work done by the forces of the system is independent of the mode of transformation, and depends only on the initial and final configurations.

Suppose that in one mode M of going from A to B more work is done than in another mode N. Let us imagine two systems precisely similar, and let the first, going from A to B by the mode M, be made to bring the other from B to A by the mode N. This will be possible, because the work consumed in going from B to A through N is equal to the work performed in going from A to B through N, since the forces in any particular position are by hypothesis independent of the *directions* in which the points of the system are moving, and therefore each element $X\,dx$ of the total work retains the same magnitude, but changes its sign, when the transformation is reversed. Hence the transformation from A to B through M will bring the second system from B to A through N, and leave an overplus of work. Let us now suppose the second system to go from A to B through M, bringing the first from B to A through N. There will again be an overplus of work. This process may be repeated for ever, and thus an inexhaustible supply of work can be obtained from permanent natural causes without any consumption of materials. The whole of experience teaches us that this is impossible. Hence the work done in going from A to B is independent of the mode of transformation.

If a system be acted on by no external forces, the work done by the forces of the system is equal to the change of kinetic energy; whence it appears that the kinetic energy T in any particular configuration depends only on the values of the variables by which the configuration is indicated, and on the initial state: in other words, we have the equation

$$T - T_0 = \phi\,(x_1, y_1, z_1, x_2, y_2, z_2, \&c.). \qquad (2)$$

It is essential to the validity of this demonstration that the work consumed by the forces of the system, in any transformation, should be equal to the work performed by them in the *same* transformation reversed. If the force acting on any point changes sign with the direction of the motion of that point, the condition of reversibility is not fulfilled. In the case of friction, for instance, so far as it is considered in Mechanics, the force changes sign with the motion, and consumes work both in the direct and reverse transformation.

The same is true of the resistance of a medium. Again, if the forces are not equal in magnitude when the points occupy the same relative positions, as in the case of the collision of imperfectly elastic bodies, work is apparently consumed without any corresponding increase of potential energy. The experiments of Joule and others have established that in all such cases the energy which seems to be lost is really preserved in the form of heat, which may be regarded as kinetic energy resulting from molecular motions not directly sensible. In applying the equation of energy we must, however, remember that in all cases such as those mentioned, as well as in some others, the conservation of energy, so far as *sensible* motion alone is concerned, does not hold good. On the other hand, if we take into account every form of energy, the conservation of energy may be considered as an absolutely universal fact of nature.

If in (2) we put $\phi = V_0 - V$, we have

$$T + V = T_0 + V_0. \tag{3}$$

We can now give an exact definition of potential energy.

The Potential Energy of a conservative system in any particular configuration is the amount of work required to bring it to that configuration against the mutual forces of the system in its passage from any chosen configuration.

The principle of the conservation of energy may then be stated thus :—

In any conservative system unacted on by external forces the sum of the kinetic and potential energies is constant.

278. **Of the ultimate Permanent Forces of Nature.**—In his Paper on the Conservation of Force (Ueber die Erhaltung der Kraft, 1847), Helmholtz observes that we must regard the forces of nature as caused by the action of portions of matter on each other, and from a mathematical point of view must consider matter as composed of an infinite number of material points. The ultimate permanent forces of nature must result, therefore, from the action of these points on each other.

If the conservation of energy hold good of these forces, the mutual forces between two material points must be in the line joining them, and be a function of the distance between them.

This proposition is proved by Helmholtz as follows:—

In this case the kinetic energy of the system composed of the two points is given by the equation

$$T = \int (X dx + Y dy + Z dz + X' dx' + Y' dy' + Z' dz') + c.$$

Since the conservation of energy holds good, T is a function of the relative position of the two points. Again, as they are points, all directions must be supposed indifferent as regards either of them considered alone. Hence their relative position must depend solely on the distance between them, and T is therefore a function of this distance r, or $T = \phi(r)$.

Equating the two expressions for T and differentiating, we have

$$X = \phi'(r)\frac{dr}{dx}, \quad Y = \phi'(r)\frac{dr}{dy}, \quad Z = \phi'(r)\frac{dr}{dz},$$

$$X' = \phi'(r)\frac{dr}{dx'}, \text{ \&c., or}$$

$$X = \phi'(r)\frac{x - x'}{r}, \quad Y = \phi'(r)\frac{y - y'}{r}, \quad Z = \phi'(r)\frac{z - z'}{r},$$

$$X' = \phi'(r)\frac{x' - x}{r}, \quad Y' = \phi'(r)\frac{y' - y}{r}, \quad Z' = \phi'(r)\frac{z' - z}{r}.$$

Hence the point x, y, z is acted on by a force $\phi'(r)$ in the direction of the line joining x', y', z' to x, y, z; and the latter point is acted on by an equal force in the opposite direction.

Conversely it is easy to see that, if two material points acted on each other with a force depending as regards magnitude on their mutual distance, but not in the direction of the line joining them, they would be capable of producing in each other an ever-increasing velocity, and of thus generating an unlimited amount of energy.

In order to bring about this result we have only to suppose the points connected by a rigid rod. The whole system would then be acted on by a constant couple.

279. **Forces which appear in the Equation of Energy.**—For any system entirely free we have obtained the equation $dT = \Sigma\,(Xdx + Ydy + Zdz)$, and have seen that the equation holds good for a system restricted in any way, provided the various constraints are replaced by equivalent forces.

If the constraints of the system consist of smooth curves or surfaces along which the points are constrained to move; of rigid bars or inextensible strings connecting the different points with each other or with any external fixed points; or in general of any connexions such that the distance between each pair of points immediately acting on each other is invariable, the whole work done by all these constraints and connexions is zero, and may therefore be omitted from the right-hand side of the equation (Arts. 121, 124).

If the potential energy (Arts. 126, 276) of any portion of the system be a function of a single variable quantity u, the work done by this part of the system in any displacement must be of the form $\lambda\delta u$, since $V = \phi(u)$, and therefore $dV = \phi'(u)\,du$.

If between any points of the system there be a connexion which is capable of being expressed by means of an equation between their coordinates, such connexion can be effected by means of material constraints of an invariable character; such as smooth fixed surfaces or curves, or rigid bars or inextensible strings.

Hence we may conclude that, *in any motion of the system, the work done by the forces replacing any connexion between the points of the system capable of being expressed by equations between their coordinates, is zero.*

A formal proof of this important proposition may be given as follows:—

If $U = 0$ be an equation between the coordinates of any points in a moving system, the forces which the corresponding constraint introduces into the system must be functions of the coordinates and of the other forces. Hence, if the latter be conservative, so are the forces caused by the constraint, which for brevity we shall refer to as the constraint U.

Again, if at any time the condition $U = 0$ be actually fulfilled, the imposition or removal of the material bonds by which the corresponding constraint is effected cannot require any expenditure of energy; since this imposition or removal does not change the position of any point of the system.

Let there be now a system S_1, which without U is conservative, and let A and B be two configurations in which the condition $U = 0$ is fulfilled; then,. as we have seen the forces replacing U are conservative, and if they consume work going from B to A, they produce work going from A to B. Let the external work W bring S_1 from A to B subject to the constraint U. Let Q be the amount of potential energy thereby produced in S_1, and E the work done by the forces replacing U; then, the whole amount of work produced is $W - Q + E$. Now let this be used in bringing S_2 (precisely similar to S_1) from B to A without U, whereby Q is produced, and in doing such an amount of other work, that S_1 may come to rest at B and S_2 at A. We may then without any expenditure of work impose the constraint U on S_2 and remove it from S_1. Things are now in precisely the same state as at starting, and in the whole process, by an expenditure of work W, we have produced work whose amount is $W + E$. Hence in any motion of the system S_1 the work E done by the forces replacing the condition $U = 0$ must be zero.

As the amount of work done by these forces in an unreversed motion cannot be influenced by the *character* of the other forces, but only by their amounts and directions, the work done by the forces replacing $U = 0$ must under any circumstances be zero.

280. **Equation of Energy in General.**—If we have a system acted on by any forces external or internal, and subject to any constraints or mutual connexions, the equation of energy assumes the form

$$T - T_0 + V - V_0 = W. \qquad (4)$$

T_0 and V_0 are the kinetic and potential energies in the initial position, T and V in the position under consideration, W is the work done by the forces external to the system, and by those internal forces which are not conservative or reversible in their character in going from the initial to the actual position.

As regards constraints and connexions, they may be divided into three classes. 1. Those producing forces whose work during any motion of the system is zero. Such connexions we have already considered; they have no effect on the equation of energy. 2. Those which are capable of alteration under the action of external forces, and such that their alteration produces or consumes a corresponding amount of potential energy. The work done by the forces replacing these constraints and connexions is included in the expression $V_0 - V$. 3. Resistances or connexions which introduce forces of a non-conservative character. Such are the friction of rough surfaces, the resistance of a medium, the forces developed by the alteration of an extensible body which does less work in its recoil than the amount required to stretch it, &c. All such forces must appear as forces in the equation of energy, and the work done by them is included in W.

281. **Virtual Velocities**.—The principle of Energy may, as we have seen, be expressed for dynamical purposes in the following form:—

The work done on a system in any interval of time by external applied forces, diminished by the work consumed in the same time by the non-conservative forces of the system, is equal to the sum of the increments of the kinetic and potential energies.

We have seen likewise that this principle holds good for a system subject to any invariable constraints or connexions internal or external as well as for a free system.

We are now able to obtain the conditions which must be fulfilled in order that any system should be in equilibrium; they can be expressed in a single statement, viz:—

In order that any system should be in equilibrium, the work done by the applied forces in any possible infinitely small displacement, diminished by the increase of the potential energy of the system, must be either negative or zero; and, if this be true for every possible infinitely small displacement, the system is in equilibrium.

The truth of this statement readily appears from the equation of energy.

A position of equilibrium is one in which if the system be placed at rest it will remain at rest. Now the system will not remain at rest if there be any possible mode of displacement, in which the united action of the internal and external forces can produce a velocity in any of the points of the system. On the other hand, if the system move from rest in any manner, it will acquire a positive kinetic energy. Hence, if there be no possible way in which it can do this, its position must be one of equilibrium.

In applying the principle of equilibrium we must regard the non-conservative forces of the system (if any) as applied forces, and introduce them with their proper signs. In the case of *actual* motion, forces of this kind always consume work, but in the case of *virtual* displacements this is not necessarily the case; *e. g.* suppose a heavy particle is placed on a rough inclined plane, and it is required to determine the condition of equilibrium. In this case we must consider the force of friction as acting upwards along the plane. If now we imagine a virtual displacement down the plane, friction will consume work; but if we imagine a displacement up the plane, friction will produce work. In the case of actual motion, whether slipping take place up or down the plane, friction will consume work.

Again it is to be observed, that if every possible set of displacements remain possible when reversed, the condition of equilibrium becomes simply that *the total work done by all the forces internal and external be zero.*

In fact, if $\Sigma P\delta p$ be negative, and P remaining unaltered, the sign of each δp be changed, $\Sigma P\delta p$ becomes positive; but this is inconsistent with equilibrium, hence $\Sigma P\delta p$ must be zero.

If we combine the principle of Virtual Velocities with D'Alembert's principle, we obtain the equation which embraces the whole theory of Kinetics,

$$\Sigma \left\{ \left(X - m\frac{d^2x}{dt^2} \right) \delta x + \left(Y - m\frac{d^2y}{dt^2} \right) \delta y + \left(Z - m\frac{d^2z}{dt^2} \right) \delta z \right\} = 0.$$

From this equation that of energy was deduced in Chapter IX. In the present Chapter we have reversed this mode of procedure.

282. **Equivalent Sets of Forces.**—Two sets of forces acting on any material system are said to be equivalent when each is capable of equilibrating the same third set.

If a set of forces acting on any system be replaced by an equivalent set, the motion of the system is unaltered.

For let P be a force of the first set, and Q one of the second, and suppose the P set to be acting. Introduce at the point where Q would act two forces Q and $-Q$. This being done for each point of the system, the motion remains undisturbed. The system is now acted on by the three sets of forces P, Q, and $-Q$; and, since the sets P and Q are equivalent, the sets P and $-Q$ are in equilibrium, and the motion is the same as if the set Q were acting alone.

In moving a system from one given position to another, equivalent sets of forces produce the same amount of work.

The motion being the same whichever set of forces is in action, the intermediate positions of the system are at each instant the same; and since the two sets of forces are equivalent, $\Sigma P dp = \Sigma Q dq$ at each instant. Hence the whole amount of work produced in one case is equal to that produced in the other.

It can be shown in like manner that the work required to move a system from one given position to another, against the action of any set of forces, is equal to that required to move it against the action of an equivalent set.

283. **Wrenches.**—A wrench in Kinetics corresponds to a twist in Kinematics.

If a rigid body be acted on by any forces, these forces can be reduced to a single force along with a couple whose plane is perpendicular to the direction of the force.

Such a system is called a *wrench about a screw*, the axis of the screw being the line of direction of the force, and the pitch of the screw being the line which is the quotient of the couple by the force. The magnitude of the force is called the intensity of the wrench.

The wrench to which a set of forces acting on a rigid body is equivalent has been termed the canonical form of the set of forces.

The canonical form of a set of forces is in general unique; for, as may be easily seen, if two wrenches be equivalent, they must either be identical or else consist of equal couples in parallel planes.

EXAMPLES.

1. A particle of mass m moves with a simple harmonic motion; determine its mean energy.

If τ and a be the periodic time and amplitude of the motion (Arts. 85, 86), and T the mean energy,

$$T = \frac{1}{\tau}\int_0^\tau \frac{mv^2}{2}\,dt = m\,\frac{\pi^2}{\tau^2}\,a^2.$$

2. If the motion of the particle m be the resultant of any number of simple harmonic motions having different periods and amplitudes, find the mean value of the energy.

If θ be an interval of time which is very great compared with the longest periodic time,

$$T = \frac{1}{\theta}\int_0^\theta \frac{mv^2}{2}\,dt = m\pi^2\,\Sigma\frac{a^2}{\tau^2}.$$

3. Determine the mean energy of a system of vibrating particles.

The rectangular components of the displacement of any particle are periodic functions of the time, and can therefore be expanded in series any one term of which is of the form $a \sin\left(\frac{2\pi}{\tau}t + \alpha\right)$. Hence,

$$T = \pi^2\Sigma m\,\frac{a^2 + b^2 + c^2}{\tau^2}.$$

4. A rigid body is acted on by a couple whose moment is Pp; determine the work done by the couple in any small motion of the body.

If $d\theta$ be the angular displacement of the body round an axis perpendicular to the plane of the couple, the work done by the couple is $Pp\,d\theta$ (see Art. 125).

5. Express the kinetic energy of a body having a fixed point in terms of the angles θ, ϕ, ψ (Art. 238), the body-axes being the principal axes at the fixed point.

As ω_1, ω_2, ω_3 are given in terms of θ, ϕ, ψ, Ex. 28, p. 316, we have, Art. 245,

$$2T = (A\sin^2\phi + B\cos^2\phi)\,\dot\theta^2 + \{(A\cos^2\phi + B\sin^2\phi)\sin^2\theta + C\cos^2\theta\}\,\dot\psi^2$$

$$+ C\dot\phi^2 + 2(B - A)\sin\theta\,\sin\phi\cos\phi\,\dot\theta\,\dot\psi + 2C\cos\theta\,\dot\phi\,\dot\psi.$$

6. If T be the kinetic energy of a body having a fixed point, and ω_x, ω_y, ω_z its angular velocities round three rectangular axes fixed in space passing

through the point, prove that $\frac{dT}{d\omega_x}, \frac{dT}{d\omega_y}, \frac{dT}{d\omega_z}$ are the moments of momentum of the body round the axes.

Let $\dot{x}, \dot{y}, \dot{z}$ be the components of the velocity of any point of the body, then

$$2T = \Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad \frac{dT}{d\omega_x} = \Sigma m\left(\dot{x}\frac{d\dot{x}}{d\omega_x} + \dot{y}\frac{d\dot{y}}{d\omega_x} + \dot{z}\frac{d\dot{z}}{d\omega_x}\right);$$

hence (Art. 235),

$$\frac{dT}{d\omega_x} = \Sigma m(y\dot{z} - z\dot{y}).$$

7. Determine the amount of energy W which must be expended on a rigid body in order to effect a given twist in opposition to a given wrench.

Let θ be the amplitude of the twist round the screw α whose pitch is p, Q the intensity of the wrench round the screw β whose pitch is q, and Ω the angle between α and β.

Take α as axis of x, and the shortest distance from α to β as axis of z, denoting its length by c. Replace the wrench at each instant by the forces X, Y, Z, passing through a point coinciding with the origin, and the couples L, M, N. Then the system X, Y, Z, L, M, N being equivalent to the wrench,

$$X = Q\cos\Omega, \quad Y = Q\sin\Omega, \quad Z = 0,$$

$$L = Qq\cos\Omega - Qc\sin\Omega, \quad M = Qq\sin\Omega + Qc\cos\Omega, \quad N = 0.$$

Hence (Ex. 4, and Art. 121),

$$dW = Qp\cos\Omega\, d\theta + (Qq\cos\Omega - Qc\sin\Omega)\,d\theta = Q\{(p+q)\cos\Omega - c\sin\Omega\}\,d\theta,$$

and therefore

$$W = Q\theta\{(p+q)\cos\Omega - c\sin\Omega\}.$$

The expression $(p+q)\cos\Omega - c\sin\Omega$ is called (Ball, *Theory of Screws*, § 13) the virtual coefficient of the pair of screws α and β.

If c be regarded as always positive, Ω is the angle through which the axis of the screw α must be turned round the axis of z in order to be codirectional with the axis of β, the positive direction of z being always from α towards β.

8. Two weights are connected by a fine inextensible string passing over a smooth pulley. The lesser hangs vertically, and the other descends a smooth inclined plane, starting without initial velocity from a point vertically under the pulley; determine how far it will descend, and state the limit of the ratio of the weights within which finite descent is possible.

If z be the height which the lesser weight W' ascends, and s the distance along the inclined plane traversed by the greater weight W, we have

$$Ws\sin i - W'z = 0,$$

and therefore

$$z = \lambda s\sin i, \quad \text{if} \quad \frac{W}{W'} = \lambda.$$

Also, if h be the vertical distance of the pulley from the inclined plane,

$$h + z = \sqrt{(h^2 + s^2 + 2hs \sin i)}.$$

Hence $s = \dfrac{2(\lambda - 1)\sin i}{1 - \lambda^2 \sin^2 i}\, h$, and in order that finite descent should be possible, $W' > W \sin i$.

9. If any system of smooth inelastic bodies collide, show that the loss of vis viva is

$$\Sigma m \left\{ (u' - u)^2 + (v' - v)^2 + (w' - w)^2 \right\},$$

where m is the mass of any particle, and u, v, w; u', v', w' are the components of its velocity before and after the shock.

If $\underset{\cdot}{X}$, &c., be the components of the impulses due to the shock, we have (Art. 202),

$$\Sigma m \{ (u' - u)\,\delta x + (v' - v)\,\delta y + (w' - w)\,\delta z \} = \Sigma (\underset{\cdot}{X}\delta x + \underset{\cdot}{Y}\delta y + \underset{\cdot}{Z}\delta z).$$

Instead of δx, δy, δz, &c., we may substitute $u'dt$, $v'dt$, $w'dt$, &c., for as these are actual displacements they must be possible: thus we have

$$\Sigma m \{ (u' - u)\,u' + (v' - v)\,v' + (w' - w)\,w' \} = \Sigma (\underset{\cdot}{X}u' + \underset{\cdot}{Y}v' + \underset{\cdot}{Z}w').$$

But x_1, y_1, z_1; x_2, y_2, z_2 being two colliding points,

$$\Sigma (\underset{\cdot}{X}u' + \underset{\cdot}{Y}v' + \underset{\cdot}{Z}w') = \underset{\cdot}{X}_1 (u_1' - u_2') + \underset{\cdot}{Y}_1 (v_1' - v_2') + \underset{\cdot}{Z}_1 (w_1' - w_2') + \&c. = 0,$$

since the direction of the impulse at each point is perpendicular to that of the relative velocity after impact. Hence

$$\Sigma m (u'^2 + v'^2 + w'^2) = \Sigma m (uu' + vv' + ww'),$$

and therefore

$$\Sigma m (u^2 + v^2 + w^2) - \Sigma m (u'^2 + v'^2 + w'^2) = \Sigma m \left\{ (u' - u)^2 + (v' - v)^2 + (w' - w)^2 \right\}.$$

If there be any set of internal impulses whose directions are perpendicular to the relative velocities assumed immediately after their action, or any set of external impulses whose directions are perpendicular to the absolute velocities assumed immediately after their action, by the points of the system at which they act, the expression just obtained for the loss of vis viva still holds good.

The impulses produced by geometrical constraints of any kind suddenly imposed fulfil the condition mentioned above.

10. If a system be put in motion by any set of impulses, every additional geometrical constraint imposed on the system diminishes its initial vis viva.

11. If a system be acted on by any set of impulses, prove that the increase of its vis viva may be expressed in the form

$$\Sigma \{ \underset{\cdot}{X}(u + u') + \underset{\cdot}{Y}(v + v') + \underset{\cdot}{Z}(w + w') \},$$

where $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$ are the components of the impulse acting at any point of the system, and u, v, w, u', v', w' the components of the velocity of this point before and after the impulsion.

Since $u_1\,dt$, $v_1\,dt$, $w_1\,dt$; $u_2\,dt$, $v_2\,dt$, $w_2\,dt$, &c., are the components of actual displacements of the points x_1, y_1, z_1; x_2, y_2, z_2, &c., they are possible displacements. The same thing is true for $u_1'\,dt$, &c. Hence any system of displacements resulting from the super-position of these two, each set being multiplied by a constant factor, is a possible system. Hence in D'Alembert's equation for impulses (Art. 202) we may substitute $u + u'$ for δx, $v + v'$ for δy, and $w + w'$ for δz, and we obtain for

$$\Sigma m\left\{(u'^2 - u^2) + (v'^2 - v^2) + (w'^2 - w^2)\right\},$$

the expression given above.

12. If in the last example $\Sigma\,(\underset{\cdot}{X}u' + \underset{\cdot}{Y}v' + \underset{\cdot}{Z}w') = 0$, prove that

$$\Sigma\,(\underset{\cdot}{X}u + \underset{\cdot}{Y}v + \underset{\cdot}{Z}w) = -\,\Sigma m\left\{(u' - u)^2 + (v' - v)^2 + (w' - w)^2\right\}.$$

13. One point of a rigid body moving freely is suddenly arrested; determine the loss of vis viva.

Let u, v, w be the components of the velocity of the point immediately before it is arrested, x, y, z, its coordinates, and $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$ the components of the impulse which it exerts on the body, the axes being the principal axes at the centre of inertia; then, $\mathfrak{T}$ being the loss of vis viva, we have $\mathfrak{T} = -\,(\underset{\cdot}{X}u + \underset{\cdot}{Y}v + \underset{\cdot}{Z}w)$ (Ex. 12). Substituting for $\underset{\cdot}{X}$, &c., their values from Ex. 49, p. 363, we obtain

$$\left\{\frac{ABC}{\mathfrak{M}} + A(B + C)x^2 + B(C + A)y^2 + C(A + B)z^2 + \mathfrak{M}Ir^4\right\}\mathfrak{T}$$

$$= ABC(u^2 + v^2 + w^2) + \mathfrak{M}r^2(BCu^2 + CAv^2 + ABw^2) + \mathfrak{M}^2Ir^2(xu + yv + zw)^2$$

$$+\,\mathfrak{M}(xu + yv + zw)\left\{AB(xu + yv - zw) + BC(yv + zw - xu) + CA(zw + xu - yv)\right\}.$$

14. If $\dot{x}$, $\dot{y}$, $\dot{z}$ be the changes in the components of the velocity of any point of a rigid body caused by any set of impulses, and $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$; π_1, π_2, π_3 the corresponding changes in the components of the velocity of the centre of inertia, and in the angular velocities of the body, prove that

$$\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \mathfrak{M}(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2) + A\pi_1^2 + B\pi_2^2 + C\pi_3^2,$$

the axes being the principal axes at the centre of inertia.

15. If any system of smooth imperfectly elastic bodies having a common coefficient of restitution collide, show that the loss of vis viva is

$$\frac{1 - e}{1 + e}\,\Sigma m\left\{(U - u)^2 + (V - v)^2 + (W - w)^2\right\},$$

where e is the coefficient of restitution, m the mass of any particle, and u, v, w; U, V, W the components of its velocity before and after the shock.

Let u', v', w' be the components of the velocity of m at the end of the first period of the impact; then, as in Ex. 9,

$$\Sigma m\{(U-u')u'+(V-v')v'+(W-w')w'\}=0,$$

and also

$$\Sigma m\{(u-u')u'+(v-v')v'+(w-w')w'\}=0.$$

From these we obtain

$$\Sigma m(u^2+v^2+w^2)-\Sigma m(U^2+V^2+W^2)$$

$$=\Sigma m\{(u'-u)^2+(v'-v)^2+(w'-w)^2-(U-u')^2-(V-v')^2-(W-w')^2\}.$$

Now the quantities $u'-u$, &c., are determined by linear equations in terms of the normal impulses R, &c., and the quantities $U-u'$, &c., are determined by linear equations in all respects the same as the former, except that eR, &c., take the place of R, &c. Hence $U-u'=e(u'-u)$, $U-u=(1+e)(u'-u)$, &c. Introducing these results into the equation above, we have

$$\Sigma m(u^2+v^2+w^2)-\Sigma m(U^2+V^2+W^2)=\frac{1-e}{1+e}\Sigma m\{(U-u)^2+(V-v)^2+(W-w)^2\}.$$

The theorems contained in Examples 9 and 15 are due to Carnot.

16. Show that the velocity v with which a fluid under a uniform pressure p escapes from a small orifice is given by the equation $v^2=2gh$, where h is the height of a column of the fluid which would produce the pressure p.

Suppose a small mass m of fluid forced through an orifice, whose section is σ, into a large volume of fluid under the pressure p. If x be the distance through which the surface σ of the fluid is pushed in by this operation, the work expended is $px\sigma$.

Hence the potential energy lost when m escapes is $px\sigma$, and this must be the kinetic energy acquired by m, therefore $\frac{mv^2}{2}=px\sigma$. Now, if ρ be the density of the fluid, $p=g\rho h$, and $m=\rho x\sigma$. Substituting, we have

$$v^2=2gh.$$

17. Determine the total energy, kinetic and potential, of a planet and satellite moving as in Ex. 7, p. 394.

If Y be the total energy

$$2Y=C\left\{n^2-\frac{\mu Mm}{C}r^{-1}\right\}+K,$$

where K is an undetermined constant, and C, &c., have the same significations as in the Example referred to.

18. A planet and a satellite move as in the last example. If with a given moment of momentum it is possible to set them moving as a rigid body, it is possible to do so in two ways, one of which requires the maximum amount of energy, and the other the minimum.

If in the equation of Ex. 7, p. 394, we substitute x for $r^{\frac{1}{2}}$, y for n, and put h for the moment of momentum, we have

$$h = Cy + \mu^{\frac{1}{2}} Mm (M + m)^{-\frac{1}{2}} x.$$

Again (Ex. 17),

$$2Y - K = Cy^2 - \mu Mm \frac{1}{x^2}.$$

By a proper selection of the independent arbitrary units of mass, length, and time we can make C, $\mu^{\frac{1}{2}} Mm (M + m)^{-\frac{1}{2}}$, and μMm each equal to unity; and we obtain thus, for the unit of mass $\frac{Mm}{M + m}$, for the unit of length $\left\{\frac{C(M + m)}{Mm}\right\}^{\frac{1}{2}}$, and for the unit of time $\left\{\frac{C^3 (M + m)}{\mu^2 M^3 m^3}\right\}^{\frac{1}{2}}$. We have then

$$h = y + x, \quad 2Y - K = y^2 - \frac{1}{x^2}.$$

If the whole system move as a rigid body, the angular velocity ω of the satellite round the centre of inertia of the system must be equal to n; but $r^3 = \mu (M + m)\, \omega^{-2}$, and in the special units adopted $\mu^{\frac{1}{2}} (M + m)^{\frac{1}{2}} = 1$; hence $x^3 = \frac{1}{y}$. Again, to have Y a maximum or a minimum,

$$\frac{d}{dx}\left\{(h - x)^2 - \frac{1}{x^2}\right\} = 0, \quad \text{or} \quad x - h + \frac{1}{x^3} = 0,$$

which is the same equation for determining x as before. Hence, if the whole system move as a rigid body, Y is a maximum or a minimum.

Let $f(x) = x^4 - hx^3 + 1 = x^3 (x - h) + 1$. If x be negative $f(x)$ is positive, and therefore the biquadratic $f(x) = 0$ has no negative root, and cannot therefore have more than two real roots, since the coefficient of x^2 is zero. If $x > h$, $f(x)$ is positive, and therefore the biquadratic has no positive root greater than h; but if x be positive and less than h, $f(x)$ may be negative, and therefore the biquadratic may have two positive roots between 0 and h. As $f'(x) = x^2 (4x - 3h)$, if the biquadratic have two real roots, one is greater than $\frac{3}{4} h$ and the other less. The greater root makes $f'(x)$, and therefore $\frac{d^2 Y}{dx^2}$, positive, and Y a minimum; the lesser root makes $\frac{d^2 Y}{dx^2}$ negative, and Y a maximum.

19. Apply the preceding examples to determine the secular effects of tidal friction on the Earth-moon system, the Moon being supposed to move in the plane of the equator.

If the Earth's radius be denoted by a, C is approximately $\frac{1}{3} Ma^2$, and $M = 82m$. Hence the unit of mass is $\frac{M}{83}$, the unit of length $5{\cdot}26a$, and the unit of time 2 hours 41 minutes. Again, in the special units, the present value of r is $11{\cdot}454$, and that of n is $0{\cdot}704$, whence x at present is $3{\cdot}384$, and $h = 4{\cdot}088$. It is plain that for this value of h the biquadratic $f(x) = 0$ has two real roots. The lesser of these, x_1, makes Y a maximum, and the greater, x_2, makes Y a minimum. Again, $f(x)$ is positive for values of x between 0 and x_1, negative for those between x_1 and x_2, and positive for those greater than x_2. As x is positive throughout, we have $\frac{1}{x^3} > y$, or $\omega > n$, when $f(x)$ is positive; and $\frac{1}{x^3} < y$, or $\omega < n$, when $f(x)$ is negative. Since the Moon's motion and the Earth's rotation are in the same direction, and since the angular velocity of the Earth's rotation exceeds that of the Moon in its orbit, the present state of things corresponds to a value of x which lies between x_1 and x_2.

We can now determine the effects of tidal friction. Since the friction resulting from the lunar tides must constantly diminish the sensible or mechanical energy of the Earth-moon system, Y must continually decrease (Art. 277). Hence, as at present $f(x)$ is negative, x must increase and y decrease until Y reaches its minimum, after which the whole system will move as if rigidly connected.

It appears accordingly that the friction caused by the lunar tides diminishes the angular velocity of the Earth, or increases the length of the day, and at the same time increases the Moon's distance and the length of the month. This process must go on till the day and month are of equal length after which the lunar tides must cease. If at any past period the Moon moved as if rigidly connected with the Earth, this must have been when Y was a maximum. Such a state of things was dynamically unstable, for the least disturbance of the rigidity of the motion would have produced tides whose friction would have diminished the energy, and caused the system to depart farther from the configuration of maximum energy. The departure from this configuration might have taken place in two ways, according as the Moon approached the Earth or receded from it. If the former event had occurred, the Moon's angular velocity in its orbit would have become greater than the angular velocity of the Earth's rotation, and the Moon must ultimately have fallen upon the Earth, as x must have decreased continually along with Y. If on the other hand the Moon had receded, the present state of things would have been reached. The value of x which makes Y a minimum lies between 4·073 and 4·074, and the corresponding value of n lies between 0·015 and 0·014. The ratios of the present value of n to these two values are 46·9 and 50·2. The present investigation would therefore lead us to conclude that, when the lunar tides cease and the day and month become equal, the length of the day will be between 46 times and 50 times its present length.

Examples 17, 18, 19, and Example 7, p. 394, are taken from a Paper by Professor G. H. Darwin, first published in the *Proceedings* of the Royal Society for 1879, and subsequently, with some alterations, in Thomson and Tait's *Natural Philosophy*, Part ii. In this Paper Mr. Darwin gives diagrams of the curves represented by the equations

$$\eta = 2Y - K = F(x), \quad x^3 y = 1, \quad h = x + y,$$

by means of which the results arrived at are exhibited to the eye.

20 The ends of a string passing over a smooth pulley are attached to two masses, of which one rests on a horizontal plane, and the other is dropped through a height h; the masses of the string and pulley being neglected, determine the loss of kinetic energy caused by the impulsive tension of the string (see Art. 210).

If m and m' be the masses, v_1 and v_2 the velocities of the dropped mass m before and after the chuck, and $\mathfrak{T}$ the loss of kinetic energy,

$$2\mathfrak{T} = m\,(v_1 - v_2)^2 + m'\,v_2^2 \text{ (Ex. 9).} \quad \text{Hence } \mathfrak{T} = \frac{m\,m'}{m+m'},\, gh.$$

21. A great number of smooth perfectly elastic particles are moving with great velocity in various directions within a rectangular parallelepiped, two of whose opposite faces are large compared with its other dimensions. If one of these faces be movable, determine the force required to keep it steady.

Let u be the velocity of one of the particles whose mass is m, and ϕ the angle which the direction of its motion makes with the normal to the face. Before striking the face the particle has a normal velocity $u\cos\phi$, and after the shock it acquires an equal normal velocity in the opposite direction. The momentum communicated to the face is therefore $2\,mu\cos\phi$. Having reached the opposite face, the particle rebounds and strikes the movable face again; the interval of time between two successive shocks against the movable face being $\dfrac{2a}{u\cos\phi}$, where a is the perpendicular distance between the faces. The whole momentum communicated to the movable face by the particle m in the time θ is therefore $\dfrac{mu^2\cos^2\phi}{a}\,\theta$, and the whole momentum M communicated by all the particles in the same time is $\dfrac{\theta}{a}\,\Sigma mu^2\cos^2\phi$.

In order to determine the value of $\Sigma mu^2\cos^2\phi$, describe a sphere of unit radius, and draw from its centre lines parallel to the directions of motion of the various particles at the beginning of the interval of time θ. Since the number of particles is very great and the direction of the motion of any one undetermined, we may assume that the energy of those particles whose directions of motion make an angle ϕ with a fixed direction is to the total energy of the particles as that portion of the surface of the sphere which lies on the cone whose semi-angle is ϕ is to the whole surface. If T be the total energy of the particles, we have then

$$\Sigma mu^2\cos^2\phi = T\int_0^\pi \cos^2\phi\sin\phi\,d\phi = \frac{2}{3}T.$$

Hence $M = \dfrac{2}{3}\dfrac{T\theta}{a}$. Now the required force F must be such as to communicate to the movable face the momentum M in the time θ, and therefore

$$F\theta = \frac{2}{3}\frac{T\theta}{a}, \text{ and } F = \frac{2}{3}\frac{T}{a}.$$

22. A number of particles move as in the last example; determine the pressure which they exert on the unit of area.

If S be the area of the movable face in the last example, and p the pressure of the particles on the unit of area, $pS = F = \dfrac{2}{3}\dfrac{T}{a}$. Hence, if v be the volume of the parallelepiped, $pv = \dfrac{2}{3}T$.

The results obtained in Ex. 21 and 22 are made use of to explain the pressure which a gas exerts against its envelope. The mode of investigation employed is due to Clausius.

23. Determine the mean kinetic energy of any system in stationary motion.

A system is said to be in stationary motion when the coordinates and velocities of its various points fluctuate within determinate finite limits.

If we integrate $\dot{x}^2 dt$ by parts, we get $\int \dot{x}^2 dt = x\dot{x} - \int x\ddot{x}\,dt$; and similar equations may be obtained corresponding to the other coordinates. Again, supposing each point of the system to be free, we have $m\ddot{x} = X$. Hence, if $\theta = t_1 - t_0$,

$$\frac{1}{\theta}\int_{t_0}^{t_1} T\,dt = \frac{1}{2\theta}\Sigma m\{x_1\dot{x}_1 + y_1\dot{y}_1 + z_1\dot{z}_1 - (x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0)\}$$
$$-\frac{1}{2\theta}\Sigma\int_{t_0}^{t_1}(Xx + Yy + Zz)\,dt.$$

If θ be made sufficiently large, the first term on the right-hand side of this equation may be neglected, and we find that the mean value of T is equal to the mean value of

$$-\tfrac{1}{2}\Sigma(Xx + Yy + Zz).$$

This latter quantity is termed by Clausius the *virial* of the system. Hence, *the mean kinetic energy is equal to the virial.* This theorem, and its application given in Ex. 25, are due to Clausius, whose investigation will be found in the *Philosophical Magazine* for August, 1870.

24. If Π be the virial of a system which is acted on by no external forces except a uniform pressure on its surface, prove that

$$\Pi = \tfrac{3}{2}pv + \frac{1}{\theta}\int_{t_0}^{t_1}\Sigma r\phi(r)\,dt,$$

where r is the distance between any two points of the system, v its volume, and p the pressure which it exerts upon the unit area of the surface of the surrounding medium or envelope.

If r be the distance between two particles of the system whose coordinates are x, y, z; x', y', z', the portion of Π due to the mutual action of these particles is the mean value of an expression of the form

$$-\phi(r)\left(\frac{x'-x}{r}x + \frac{y'-y}{r}y + \frac{z'-z}{r}z\right) - \phi(r)\left(\frac{x-x'}{r}x' + \frac{y-y'}{r}y' + \frac{z-z'}{r}z'\right),$$

or $\quad r\phi(r)$. (Art. 278.)

Again, if dS be an element of the bounding surface of the system the direction cosines of whose normal are l, m, n, the part of Π due to the external pressure is

$$\tfrac{1}{2}p\iint(xl + ym + zn)\,dS = \tfrac{1}{2}p\left\{\iint x\,dy\,dz + \iint y\,dz\,dx + \iint z\,dx\,dy\right\} = \tfrac{3}{2}pv.$$

Hence
$$\Pi = \tfrac{3}{2}pv + \frac{1}{\theta}\int_{t_0}^{t_1}\Sigma r\phi(r)\,dt.$$

25. Determine the pressure of an enclosed gas in terms of its volume and the mean kinetic energy of translation of its molecules.

A gas may be regarded as composed of a number of molecules which exert no action on each other except when in contact. If the gas be enclosed in an envelope, and its condition remain unaltered, its molecules must be in stationary motion. Hence, if T' be the mean value of that part of the kinetic energy which results from the velocities of the centres of inertia of the molecules, and Π the corresponding virial, we have $T' = \Pi$; but $\Pi = \tfrac{3}{2}pv$ (Ex. 24), since the other term of Π may in this case be neglected. Accordingly $pv = \tfrac{2}{3}T'$.

284. **General Equations of Motion for any System.**—The general equations of motion for any system are obtained in precisely the same manner as the general equations of equilibrium.

If $F = 0$, $G = 0$, $H = 0$, &c., are the equations of condition representing the connexions and restraints, we have

$$\frac{dF}{dx_1}\,\delta x_1 + \frac{dF}{dy_1}\,\delta y_1 + \frac{dF}{dz_1}\,\delta z_1 + \frac{dF}{dx_2}\,\delta x_2 + \&c. = 0.$$

$$\frac{dG}{dx_1}\,\delta x_1 + \&c. = 0,\ \frac{dH}{dx_1}\,\delta x_1 + \&c. = 0,\ \&c.$$

Multiply the first by λ, the second by μ, the third by ν, &c., and add to D'Alembert's Equation, and we obtain

$$\left(X_1 - m_1\frac{d^2x_1}{dt^2} + \lambda\frac{dF}{dx_1} + \mu\frac{dG}{dx_1} + \nu\frac{dH}{dx_1} + \&c.\right)\delta x_1 + \&c. = 0.$$

If there be n equations of condition we can assign to λ, μ, ν, &c., such values as to make the coefficients of the first n displacements in the above equation vanish. By means of these displacements we can satisfy the n equations $\delta F = 0$, $\delta G = 0$, &c. The remaining displacements are then entirely unrestricted, and their coefficients in the general equation above must therefore be each zero, and we have for the equations of motion

$$m_1\frac{d^2x_1}{dt^2} = X_1 + \lambda\frac{dF}{dx_1} + \mu\frac{dG}{dx_1} + \nu\frac{dH}{dx_1} + \&c.,$$

$$m_1\frac{d^2y_1}{dt^2} = Y_1 + \lambda\frac{dF}{dy_1} + \mu\frac{dG}{dy_1} + \nu\frac{dH}{dy_1} + \&c.,$$

$$m_1\frac{d^2z_1}{dt^2} = Z_1 + \lambda\frac{dF}{dz_1} + \mu\frac{dG}{dz_1} + \nu\frac{dH}{dz_1} + \&c.,$$

$$m_2\frac{d^2x_2}{dt^2} = X_2 + \lambda\frac{dF}{dx_2} + \mu\frac{dG}{dx_2} + \nu\frac{dH}{dx_2} + \&c.,$$

$$\&c., \qquad \&c., \qquad \&c.$$

If from these equations we seek to obtain the Equation of Energy, we multiply the first by dx_1, the second by dy_1, &c., and add, and we have

$$dT = \Sigma\,(Xdx + Ydy + Zdz) + \lambda\left(\frac{dF}{dx_1}dx_1 + \frac{dF}{dy_1}dy_1 + \&c.\right) + \&c.$$

Now, if the equation $F = 0$ involve only the coordinates of the various points,

$$\frac{dF}{dx_1}dx_1 + \frac{dF}{dy_1}dy_1 + \&c. = dF = 0,$$

and the condition expressed by the equation $F = 0$ has no effect on the kinetic energy.

This result was obtained from first principles in Art. 279, and by its means the Equation of Virtual Velocities in its usual form was deduced from the Equation of Energy.

285. **Equation of Energy when Equations of Condition Involve the Time Explicitly.**—If the equation $F = 0$ involve the time explicitly, the work done in any actual motion of the system by the forces capable of replacing the condition $F = 0$ need not be zero. In a virtual displacement the work done by these forces must still be zero, because in such a displacement no lapse of time is supposed to take place. So far, therefore, as the equation of virtual velocities is concerned, t must be considered constant in the equation $F = 0$, and the virtual displacements must fulfil the condition

$$\frac{dF}{dx_1}\delta x_1 + \frac{dF}{dy_1}\delta y_1 + \frac{dF}{dz_1}\delta z_1 + \frac{dF}{dx_2}\delta x_2 + \&c. = 0.$$

The actual displacements on the other hand fulfil the condition

$$\frac{dF}{dx_1}dx_1 + \frac{dF}{dy_1}dy_1 + \frac{dF}{dz_1}dz_1 + \frac{dF}{dx_2}dx_2 + \&c. + \left(\frac{dF}{dt}\right)dt = 0.$$

Hence in this case the Equation of Energy becomes

$$dT = \Sigma\,(Xdx + Ydy + Zdz) - \lambda\left(\frac{dF}{dt}\right)dt - \mu\left(\frac{dG}{dt}\right)dt - \&c. \quad (5)$$

286. **Similar Mechanical Systems.**—Two systems are geometrically similar when each line of the one is equal to the corresponding line of the other multiplied by the same constant.

Similar Mechanical systems are not only geometrically similar, but have also a similar distribution of mass, and a similar distribution of force, and work in a similar manner; *i. e.* each mass of the one is equal to the corresponding mass of the other multiplied by a constant, each force of the one is equal to the corresponding force of the other multiplied by a constant, and the systems are always geometrically similar at instants of time whose intervals from two fixed epochs are in a constant ratio.

If x be a coordinate of a point in the first system, m a mass, X a force, and t an interval of time; and if x', m', X', t' be the corresponding quantities for the second system; we have the equations $x' = lx$, $m' = \mu m$, $X' = \lambda X$, $t' = \nu t$.

Hence
$$\Sigma m' \left(\frac{d^2x'}{dt'^2}\,\delta x' + \frac{d^2y'}{dt'^2}\,\delta y' + \frac{d^2z'}{dt'^2}\,\delta z' \right)$$
$$= \frac{\mu l^2}{\nu^2}\,\Sigma m \left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z \right),$$

and
$$\Sigma\,(X'\delta x' + Y'\delta y' + Z'\delta z') = \lambda l \Sigma\,(X\delta x + Y\delta y + Z\delta z).$$

In order then that D'Alembert's equation should hold good for each system we must have $\mu l = \lambda\nu^2$.

This equation of condition may be put into another form by expressing ν in terms of the ratio of the corresponding velocities in the two systems. If we denote this ratio by a, we have $\frac{dx'}{dt'} = a\,\frac{dx}{dt}$, but also, $\frac{dx'}{dt'} = \frac{l}{\nu}\frac{dx}{dt}$; therefore $l = a\nu$, and the equation of condition becomes $\lambda l = \mu a^2$.

If, as is generally the case, weight be one of the moving forces in both systems, $\lambda = \mu$, whence $a^2 = l$, or the velocity in each system must be proportional to the square root of its linear dimensions.

287. **Equations of Motion in Generalized Coordinates.**—If the position of a system is determined at each instant by a certain set of independent quantities, which may be termed *coordinates*, and be denoted by θ, ϕ, ψ, &c.; the Cartesian coordinates of each point of the system are expressible in terms of these new variables, so that x_1, y_1, z_1, x_2, y_2, z_2, &c., are functions of θ, ϕ, ψ, &c. If the system receive a displacement $\delta\theta$, the work done against the forces of inertia must, in a conservative system, be equal to the loss of potential energy; and in any system, conservative or not, must be equal to the work done by the moving forces (Arts. 201, 281).

Hence in any system, if we put, in accordance with Newton's notation,

$$\dot{x} \text{ for } \frac{dx}{dt}, \quad \ddot{x} \text{ for } \frac{d^2x}{dt^2}; \quad \dot{y} \text{ for } \frac{dy}{dt}, \quad y \text{ for } \frac{d^2y}{dt^2}, \text{ \&c.,}$$

we have

$$\left\{\Sigma m\left(\ddot{x}\frac{dx}{d\theta} + \ddot{y}\frac{dy}{d\theta} + \ddot{z}\frac{dz}{d\theta}\right)\right\}\delta\theta = \left\{\Sigma\left(X\frac{dx}{d\theta} + Y\frac{dy}{d\theta} + Z\frac{dz}{d\theta}\right)\right\}\delta\theta.$$

The right-hand side of this equation may be denoted by $\Theta\delta\theta$; similar equations may be obtained for displacements corresponding to the other variables.

Since x, &c., are functions of t, in virtue of being functions of θ, ϕ, ψ, &c., we have

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\theta}\dot{\theta} + \frac{dx}{d\phi}\dot{\phi} + \frac{dx}{d\psi}\dot{\psi} + \text{\&c.};$$

therefore $$\frac{d\dot{x}}{d\dot{\theta}} = \frac{dx}{d\theta}, \quad \frac{d\dot{x}}{d\dot{\phi}} = \frac{dx}{d\phi}, \text{ \&c.};$$

and also

$$\frac{d\dot{x}}{d\theta} = \frac{d^2x}{d\theta^2}\dot{\theta} + \frac{d^2x}{d\theta\, d\phi}\dot{\phi} + \frac{d^2x}{d\theta\, d\psi}\dot{\psi} + \text{\&c.} = \frac{d}{dt}\left(\frac{dx}{d\theta}\right).$$

Again, $\frac{1}{2}\frac{d(\dot{x}^2)}{d\dot{\theta}} = \dot{x}\frac{d\dot{x}}{d\dot{\theta}} = \dot{x}\frac{dx}{d\theta}$, and therefore

$$\frac{1}{2}\frac{d}{dt}\frac{d(\dot{x}^2)}{d\dot{\theta}} = \ddot{x}\frac{dx}{d\theta} + \dot{x}\frac{d}{dt}\left(\frac{dx}{d\theta}\right) = \ddot{x}\frac{dx}{d\theta} + \dot{x}\frac{d\dot{x}}{d\theta} = \ddot{x}\frac{dx}{d\theta} + \frac{1}{2}\frac{d(\dot{x}^2)}{d\theta}.$$

Hence $$\ddot{x}\frac{dx}{d\theta} = \frac{1}{2}\frac{d}{dt}\frac{d(\dot{x}^2)}{d\dot{\theta}} - \frac{1}{2}\frac{d(x^2)}{d\theta}. \tag{6}$$

Now T, the kinetic energy of the system, is

$$\tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2);$$

hence, multiplying each equation similar to (6) by the corresponding mass and adding, we obtain

$$\Sigma m\left(\ddot{x}\frac{dx}{d\theta} + \ddot{y}\frac{dy}{d\theta} + \ddot{z}\frac{dz}{d\theta}\right) = \frac{d}{dt}\frac{dT}{d\dot{\theta}} - \frac{dT}{d\theta}.$$

Similar equations hold good for the other coordinates ϕ, &c.; hence the equations of motion for any system are

$$\left.\begin{array}{c} \dfrac{d}{dt}\dfrac{dT}{d\dot{\theta}} - \dfrac{dT}{d\theta} = \Theta \\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\phi}} - \dfrac{dT}{d\phi} = \Phi \\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\psi}} - \dfrac{dT}{d\psi} = \Psi \\[2ex] \&c. \end{array}\right\}. \tag{7}$$

The quantities Θ, Φ, Ψ, &c., are called the generalized components of the forces which act on the system.

For a conservative system, equations (7) become

$$\left.\begin{array}{l} \dfrac{d}{dt}\dfrac{dT}{d\dot{\theta}} - \dfrac{dT}{d\theta} + \dfrac{dV}{d\theta} = 0 \\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\phi}} - \dfrac{dT}{d\phi} + \dfrac{dV}{d\phi} = 0 \\[2ex] \dfrac{d}{dt}\dfrac{dT}{d\dot{\psi}} - \dfrac{dT}{d\psi} + \dfrac{dV}{d\psi} = 0 \\[2ex] \qquad \&c. \end{array}\right\}, \qquad (8)$$

where V is the potential energy of the system.

These equations were first given by Lagrange, and are known as *Lagrange's Equations of Motion in Generalized Coordinates.*

As observed by Thomson and Tait, the proof given above shows that Lagrange's Equations hold good when the time appears explicitely in the equations which determine the Cartesian coordinates in terms of the generalized coordinates.

In fact, in this case $x = f(\theta, \phi, \psi, \&c., t)$, &c.; $\dot{x}$ then contains the additional term $\dfrac{df}{dt}$; but the equations

$$\frac{d\dot{x}}{d\dot{\theta}} = \frac{dx}{d\theta}, \quad \frac{d\dot{x}}{d\theta} = \frac{d}{dt}\left(\frac{dx}{d\theta}\right),$$

still hold good, so that the proof given above remains valid.

V in Art. 260 represents the force function. In the present Article the same letter is used to signify the potential energy, which is equal to the force function with its sign reversed. Whether the system be conservative or not, the only forces which enter into Θ, &c. are those which appear in the equation of energy.

288. **Lagrange's Equations for Impulses.**—If immediately before and immediately after the action of any set of impulses, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$; $\dot{x}$, $\dot{y}$, $\dot{z}$, be the components of the

velocity of the point x, y, z, and if $\underset{\cdot}{X}, \underset{\cdot}{Y}, \underset{\cdot}{Z}$, be the components of the impulse at this point, we have, Art. 202,

$$\Sigma m\left\{(\dot{x} - \dot{x}_0)\frac{dx}{d\theta} + (\dot{y} - \dot{y}_0)\frac{dy}{d\theta} + (\dot{z} - \dot{z}_0)\frac{dz}{d\theta}\right\}$$
$$= \Sigma\left(\underset{\cdot}{X}\frac{dx}{d\theta} + \underset{\cdot}{Y}\frac{dy}{d\theta} + \underset{\cdot}{Z}\frac{dz}{d\theta}\right).$$

The right-hand side of this equation may be denoted by $\underset{\cdot}{\Theta}$; and since (Art. 287) $\frac{1}{2}\frac{d(\dot{x}^2)}{d\dot{\theta}} = \dot{x}\frac{dx}{d\theta}$,

$$\frac{dT}{d\dot{\theta}} = \Sigma m\left(\dot{x}\frac{dx}{d\theta} + \dot{y}\frac{dy}{d\theta} + \dot{z}\frac{dz}{d\theta}\right).$$

Now $\frac{dx}{d\theta}$, &c., are the same immediately before and immediately after the impulsion; hence, if T_0 and T be the corresponding kinetic energies, we obtain the equations

$$\left.\begin{array}{c} \dfrac{dT}{d\dot{\theta}} - \dfrac{dT_0}{d\dot{\theta}_0} = \underset{\cdot}{\Theta} \\[2ex] \dfrac{dT}{d\dot{\phi}} - \dfrac{dT_0}{d\dot{\phi}_0} = \underset{\cdot}{\Phi} \\[2ex] \&c. \end{array}\right\}. \qquad (9)$$

EXAMPLES.

1. Determine in polar coordinates the equations of motion of a particle which moves freely in a fixed plane.

Here $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, whence

$$\frac{d}{dt}\frac{dT}{d\dot{r}} - \frac{dT}{dr} = m\ddot{r} - mr\dot{\theta}^2, \quad \frac{d}{dt}\frac{dT}{d\dot{\theta}} - \frac{dT}{d\theta} = m\frac{d}{dt}(r^2\dot{\theta}),$$

and the equations of motion are the same as (9) and (10), Art. 28.

2. Determine in polar coordinates the general equations of motion of a free particle.

$$T = \tfrac{1}{2}m\{\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\},$$

and the equations of motion are

$$m\{\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\} = R,$$

$$m\left\{\frac{d}{dt}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\,\dot{\phi}^2\right\} = Pr, \qquad m\frac{d}{dt}(r^2\sin^2\theta\,\dot{\phi}) = Qr\sin\theta,$$

where R, P, and Q are the components of the force acting on the particle, along the radius vector from the origin, perpendicular to the radius vector in the meridian of the particle, and at right angles to these two directions.

3. Prove Euler's equations for a body having a fixed point.

The expression for T in terms of θ, ϕ, ψ, the body-axes being the principal axes at the fixed point, is given in Ex. 5, p. 405. Hence

$$\frac{d}{dt}\{C\dot{\phi} + C\dot{\psi}\cos\theta\} - (A - B)\sin\phi\cos\phi\,(\dot{\theta}^2 - \dot{\psi}^2\sin^2\theta)$$
$$+ (A - B)\sin\theta\,(\cos^2\phi - \sin^2\phi)\,\dot{\theta}\,\dot{\psi} = \Phi.$$

If we substitute ω_3 for $\dot{\phi} + \dot{\psi}\cos\theta$ ((9), Art. 238), and then make $\theta = \phi = \frac{1}{2}\pi$, we have (Art. 238),

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\,\omega_2 = \Phi = N. \quad \text{(Ex. 4, p. 405).}$$

4. Generalize Euler's equations for the case in which the body-axes are not principal axes.

In the case of a body having a fixed point, T is a quadratic function of ω_1, ω_2, ω_3, with constant coefficients (Art. 252). Hence by Ex. 28, p. 316,

$$\frac{dT}{d\dot{\phi}} = \frac{dT}{d\omega_1}\frac{d\omega_1}{d\dot{\phi}} + \frac{dT}{d\omega_2}\frac{d\omega_2}{d\dot{\phi}} + \frac{dT}{d\omega_3}\frac{d\omega_3}{d\dot{\phi}} = \frac{dT}{d\omega_3},$$

$$\frac{dT}{d\phi} = \frac{dT}{d\omega_1}\frac{d\omega_1}{d\phi} + \frac{dT}{d\omega_2}\frac{d\omega_2}{d\phi} + \frac{dT}{d\omega_3}\frac{d\omega_3}{d\phi} = \frac{dT}{d\omega_1}\omega_2 - \frac{dT}{d\omega_2}\omega_1;$$

and we have

$$\frac{d}{dt}\left(\frac{dT}{d\omega_3}\right) - \frac{dT}{d\omega_1}\omega_2 + \frac{dT}{d\omega_2}\omega_1 = \Phi = N.$$

5 Two particles m and m' are connected by an inextensible string passing through a smooth hole at the edge of a smooth horizontal table on which m rests; determine the equations of motion of the particles, and the tension of the string.

Let r and θ be the polar coordinates of m with respect to the hole as origin; then

$$2T = (m + m')\,\dot{r}^2 + mr^2\,\dot{\theta}^2,$$

and the equations of motion are

$$(m + m')\,\ddot{r} - mr\dot{\theta}^2 = -m'g, \quad \frac{d}{dt}(mr^2\,\dot{\theta}) = 0.$$

If τ be the tension of the string, and h the value of $mr^2\dot{\theta}$, we have

$$m\ddot{r} - mr\dot{\theta}^2 = -\tau \text{ (Ex. 1)},$$

whence

$$\tau = \frac{mm'}{m+m'}\left(g + \frac{h^2}{m^2r^3}\right).$$

6. A smooth particle descends the upper edge of a thin vertical lamina, which is capable of sliding freely down a smooth inclined plane with which its whole lower edge is in contact. If the plane of the lamina be perpendicular to the intersection of the inclined plane with the horizon, and the particle and lamina start from rest, determine their position at any time.

Let x be the distance at any time of a point in the base of the lamina from its initial position, ξ the distance which the particle has moved along the edge of the lamina, α the angle which this edge makes with the inclined plane, β the inclination of the latter, m the mass of the particle, and M that of the lamina.

The kinetic energy of the lamina at any time is $\frac{1}{2}M\dot{x}^2$, and that of the particle is

$$\tfrac{1}{2}m\{(\dot{x} + \dot{\xi}\cos\alpha)^2 + \dot{\xi}^2\sin^2\alpha\}.$$

Hence

$$2T = (M+m)\dot{x}^2 + m\dot{\xi}^2 + 2m\dot{x}\dot{\xi}\cos\alpha.$$

Again, $$-V = Mgx\sin\beta + mg\{x\sin\beta + \xi\sin(\alpha+\beta)\},$$

and therefore the equations of motion are

$$(M+m)\ddot{x} + m\ddot{\xi}\cos\alpha = (M+m)g\sin\beta, \quad m(\ddot{\xi} + \ddot{x}\cos\alpha) = mg\sin(\alpha+\beta),$$

whence

$$x = \tfrac{1}{2}gt^2\left\{\sin\beta - \frac{m\sin\alpha\cos\alpha\cos\beta}{M + m\sin^2\alpha}\right\}, \quad \xi = \tfrac{1}{2}gt^2\,\frac{(M+m)\sin\alpha\cos\beta}{M + m\sin^2\alpha}.$$

7. A particle descends from rest along one face of a smooth triangular prism which is supported by a smooth horizontal plane. The initial position of the particle lies in the vertical plane containing the centre of inertia of the prism and perpendicular to its edge; determine the motion.

Let x be the horizontal coordinate, in the vertical plane in which the particle moves, of the centre of inertia of the prism, M its mass, m that of the particle, ξ the distance it has moved at any time along the face of the prism, and α the angle which this face makes with the horizontal plane; then

$$2T = (M+m)\dot{x}^2 + m\dot{\xi}^2 + 2m\dot{x}\dot{\xi}\cos\alpha, \quad V = -mg\xi\sin\alpha;$$

and the equations of motion are

$$(M+m)\ddot{x} + m\ddot{\xi}\cos\alpha = 0, \quad m\ddot{\xi} + m\ddot{x}\cos\alpha = mg\sin\alpha.$$

Hence, as the particle starts from rest,

$$(M+m)\dot{x} = -m\dot{\xi}\cos\alpha, \quad (M+m\sin^2\alpha)\ddot{\xi} = (M+m)\,g\sin\alpha.$$

The student will observe that if T were expressed by means of the first of these equations as a function of $\dot{\xi}$ alone, and treated as such, the second equation would be obtained directly as Lagrange's equation. The importance of this remark will appear from Examples 9, 10, 11, 12.

8. In the preceding example, if the face of the prism down which the particle descends be rough, determine the equations of motion.

The force of friction tends merely to stop the relative motion of the particle and prism; hence, F being this force,

$$F\delta f = -\mu mg\cos\alpha\,\delta\xi.$$

The equations of motion are therefore

$$(M+m)\ddot{x} + m\ddot{\xi}\cos\alpha = 0, \quad m\ddot{\xi} + m\ddot{x}\cos\alpha = mg\,(\sin\alpha - \mu\cos\alpha).$$

9. If there be no force tending to alter one or more of the independent variables by which the position of a system is defined; if the expression for the kinetic energy of the system does not contain these variables, but only their differential coefficients; and if the system start from rest; then T may be expressed as a function of the other independent variables and their differential coefficients, and be treated as if these latter variables completely defined the position of the system.

Let ξ be one of the independent variables satisfying the conditions supposed; then, as there is no force tending to alter ξ,

$$\frac{d}{dt}\,\frac{dT}{d\dot{\xi}} - \frac{dT}{d\xi} = 0; \text{ but } \frac{dT}{d\xi} = 0, \text{ and therefore } \frac{dT}{d\dot{\xi}} = \text{constant};$$

but, as the system starts from rest and T is a homogeneous quadratic function of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, $\dot{\xi}$, &c., this constant must be zero; hence $\dfrac{dT}{d\dot{\xi}} = 0$. In like manner, if η be another variable satisfying the same conditions, we have $\dfrac{dT}{d\dot{\eta}} = 0$, and so on. From the linear equations $\dfrac{dT}{d\dot{\xi}} = 0$, $\dfrac{dT}{d\dot{\eta}} = 0$, &c., $\dot{\xi}$, $\dot{\eta}$, &c., can be determined in terms of the remaining differential coefficients $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, &c. Thus T becomes a function of θ, ϕ, ψ, &c., and of their differential coefficients, or

$$T = F(\theta, \phi, \psi, \text{\&c.}, \dot{\theta}, \dot{\phi}, \dot{\psi}, \text{\&c.}).$$

If now we regard θ, ϕ, ψ, &c., as completely defining the position of the system, Lagrange's equations are

$$\frac{d}{dt}\frac{dF}{d\dot{\theta}} - \frac{dF}{d\theta} = \Theta, \text{ \&c.};$$

but these equations are true, for

$$\frac{dF}{d\dot{\theta}} = \frac{dT}{d\dot{\theta}} + \frac{dT}{d\dot{\xi}}\frac{d\dot{\xi}}{d\dot{\theta}} + \frac{dT}{d\dot{\eta}}\frac{d\dot{\eta}}{d\dot{\theta}} + \text{\&c.},$$

$$\frac{dF}{d\theta} = \frac{dT}{d\theta} + \frac{dT}{d\dot{\xi}}\frac{d\dot{\xi}}{d\theta} + \frac{dT}{d\dot{\eta}}\frac{d\dot{\eta}}{d\theta} + \text{\&c.};$$

whence, as $\frac{dT}{d\dot{\xi}} = 0$, $\frac{dT}{d\dot{\eta}} = 0$, &c., we have $\frac{dF}{d\dot{\theta}} = \frac{dT}{d\dot{\theta}}$, $\frac{dF}{d\theta} = \frac{dT}{d\theta}$, &c.

10. A system is moving in any way; determine the set of impulses which would give to each part of the system, if at rest, its actual velocity.

Let $\underset{\cdot}{\Theta}$, $\underset{\cdot}{\Phi}$, &c., be the required impulses corresponding to the independent variables θ, ϕ, &c., and let T be the actual kinetic energy of the system; then

$$\Theta = \frac{dT}{d\dot{\theta}}, \quad \underset{\cdot}{\Phi} = \frac{dT}{d\dot{\phi}}, \text{ \&c. (Art. 288).}$$

11. A sphere, having no motion of rotation, and under the action of a force passing through its centre of inertia, moves through a liquid extending indefinitely in all directions on one side of an infinite plane: the liquid being originally at rest, and not acted on by any force, determine the form of the equations of motion of the sphere.

Let the origin be anywhere in the fixed plane, the axis of x being at right angles to that plane; and let x, y, z be the coordinates of the centre of the sphere at any time, and ξ a coordinate of any particle of the liquid, which may be defined as matter which is incompressible and devoid of resistance to change of shape.

If T be the kinetic energy of the whole system, we have $\frac{dT}{d\dot{\xi}} = C$, since there is no force acting on the liquid; but as the liquid was originally at rest, and no impulse was imparted to it, $C = 0$. Hence T is a function of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ (Ex. 9). Again, the motion of the system at any instant could be produced from rest by placing the sphere in its actual position, and giving it an impulse sufficient to impart to it its actual velocity, since the impulses which should be given to the liquid particles are zero (Ex. 10). Hence, as the initial circumstances are unaltered by changing the values of y and z, T is a function of x, $\dot{x}$, $\dot{y}$, $\dot{z}$. Again, a change in the sign of $\dot{y}$ or $\dot{z}$ can make no change in the value of T, which must therefore be of the form $\frac{1}{2}\{P\dot{x}^2 + Q(\dot{y}^2 + \dot{z}^2)\}$, since the coefficients of $\dot{x}\dot{y}$, $\dot{y}\dot{z}$, $\dot{z}\dot{x}$ must be zero.

The equations of motion are then

$$Q\ddot{y} + \frac{dQ}{dx}\dot{x}\dot{y} = Y, \quad Q\ddot{z} + \frac{dQ}{dx}\dot{x}\dot{z} = Z;$$

$$P\ddot{x} + \tfrac{1}{2}\left\{\frac{dP}{dx}\dot{x}^2 - \frac{dQ}{dx}(\dot{y}^2 + \dot{z}^2)\right\} = X.$$

12. Prove that a sphere projected through a liquid parallel to an infinite plane boundary is attracted towards the boundary. Show also that if projected perpendicularly from the boundary it is at first accelerated, and then tends towards a constant velocity.

Initial circumstances in Ex. 11 are altered in the same manner, whether we suppose introduced into the liquid a second bounding plane parallel to the first, and between it and the sphere, or suppose the sphere placed initially nearer the original bounding plane. Hence a diminution of the initial value of x is equivalent to the introduction of additional geometrical constraints into the system. From this it follows, by Ex. 10, p. 407, that, if $x' < x$, and $P'\dot{x}' = P\dot{x}$, the value of $P\dot{x}^2$ must exceed that of $P'\dot{x}'^2$, and therefore $\dot{x}' < \dot{x}$, and $P' > P$, or P decreases as x increases. Similar reasoning can be applied to Q. If x be infinite, or the liquid unbounded in every direction, P and Q are constants.

The statements made in the enunciation of this example follow then immediately from the equations of Ex. 11, by making X and Y zero.

Examples 9, 10, 11, 12, are taken, with some slight modifications, from Thomson and Tait (*Natural Philosophy*), and illustrate what these authors have termed *Ignoration of Coordinates.*

CHAPTER XIII.

APPLICATION OF THE THEORY OF ENERGY TO THERMODYNAMICS.

289. **Mechanical Equivalent of Heat.**—The experiments of Joule and others have shown that whenever sensible kinetic energy disappears without a corresponding increase of potential energy, an amount of heat is produced proportional to the quantity of sensible kinetic energy which has disappeared.

A similar result takes place in all cases in which work is expended without producing a corresponding increase of energy; and, conversely, a definite amount of heat can be transformed into a definite amount of work.

The number of units of work which the unit of heat can perform is called the *mechanical equivalent of heat*, and may be designated by the letter J. If the quantity of heat required to raise the temperature of the unit mass of water from 0° to 1° Centigrade be taken as the unit of heat, and the amount of work expended in lifting the unit mass through a height of one metre as the unit of work, the value of J is 424. In English units, *i.e.* if a foot be taken as the unit of length, and temperature be estimated by Fahrenheit's thermometric scale, the value of J is 772.

If Q be the quantity of heat which has been imparted to a body; U its total energy, kinetic and potential; W the work done by the body against external forces; and ΔQ, &c., the increments of these quantities at any time reckoned from the same instant, the experiments of Joule, already mentioned, conduct to the equation

$$J\Delta Q = \Delta U + \Delta W. \qquad (1)$$

If we desire to give this equation a purely theoretical basis, we have only to assume that heat is energy resulting from molecular motions, and that the principle of the conservation of energy holds good.

If we seek to determine J from Equation (1) by measuring the amount of ΔQ, &c. in any particular case, we are met by the difficulty that the value of ΔU is in general unknown. This difficulty can be got over by bringing back the body which is being experimented on to its initial condition: ΔU is then zero, and we have $J\Delta Q = \Delta W$.

When J is known, quantities of heat can be expressed in work units; and if this mode of expressing Q be adopted, (1) takes the simpler form

$$\Delta Q = \Delta U + \Delta W. \tag{2}$$

290. **Equation of Energy.**—The equation of the preceding Article is one of the two fundamental equations of Thermodynamics, and may be called the Equation of Energy.

In order to arrive at the results which may be deduced from this equation, the body under consideration is supposed to pass continuously from one state to another, in consequence of changes in its temperature and in the pressure which the unit area of its surface exerts against the surrounding medium, this pressure being supposed the same at all points of the surface.

In Thermodynamics we are not, in general, concerned with the kinetic energy of sensible motion. In fact, the apparent effect of heat on a body is to raise its temperature, or to change its condition, or to cause it to do external work. Hence the body is, in general, supposed to be at rest in the ordinary sense, and its kinetic energy is exhibited in the form not of sensible motion, but of those molecular motions on which temperature depends. This being understood, we see that the total energy of the body at any time is a function of two independent variables—the temperature t, and the pressure p.

As the volume v of the unit of mass depends on p and t, we may take as independent variables any two of the three quantities t, p, and v.

If the work done by the body against external forces be, as is usually the case, the work which it does in consequence of its expansion against the pressure on its surface, and if we consider merely the unit of mass, it is easily seen that $dW = p\,dv$, and hence we have from (1) the equation

$$JdQ = \frac{dU}{dt}dt + \frac{dU}{dv}dv + p\,dv. \tag{3}$$

291. **Specific Heat.**—The quantity of heat which must be imparted to the unit of mass of a homogeneous substance to raise its temperature 1°, is called its *specific heat*, and is equal to the limit of $\frac{\Delta Q}{\Delta t}$.

The specific heat in general depends on the external work accomplished by the body, and is indeterminate, except the relation between the variations of the independent variables be assigned. If, however, the temperature of the body be raised under the condition that its volume remains constant, or under the condition that the pressure on its surface remains constant, the specific heat is a definite function of the variables on which the state of the body depends.

The specific heat at constant volume may be designated by C_v, and that at constant pressure by C_p. We have then, from (3),

$$JC_v = \left(\frac{dU}{dt}\right)_v, \quad JC_p = \left(\frac{dU}{dt}\right)_p + p\frac{dv}{dt},$$

where $\left(\frac{dU}{dt}\right)_v$ indicates the differential coefficient of U with respect to t under the hypothesis that v is constant, and $\left(\frac{dU}{dt}\right)_p$ has a similar signification in reference to p, and where in the equation for C_p we regard v as a function of p and t.

For practical purposes, when great accuracy is not required, no distinction is made in the case of solid and liquid bodies between the two specific heats, and the specific heat for each body is assumed to be an absolute constant.

292. **Perfect Gas.**—In the case of a perfect gas the volume v of the unit of mass is connected with the pressure p, and the temperature t by an equation of the form

$$vp = v_0 p_0 (1 + at),$$

where v_0 is the volume corresponding to the pressure p_0 and the temperature zero, and a is a constant which is the same for all gases, its value being $\frac{1}{273}$ when temperatures are counted on the Centigrade thermometer. If the zero of temperature be taken at $-273°$ C., and temperature reckoned from this origin be denoted by T, we have, putting R for av_0p_0,

$$vp = RT. \tag{4}$$

The experiments of Joule and Thomson have shown that if the volume of a gas vary without any heat being imparted or abstracted, the temperature remains constant, provided no external work is done. If now in (2) we make $\Delta Q = 0$, and $\Delta W = 0$, we have $\Delta U = 0$; hence it appears, that if the temperature of a gas remains invariable, so likewise does the internal energy, which is therefore a function of the temperature alone.

The experiments of Regnault have shown that the specific heat of a gas at constant pressure is independent of the pressure, being a constant for each gas. From this it follows (see Ex. 3) that the specific heat at constant volume is likewise a constant. In the case of a perfect gas, equation (3) accordingly becomes

$$JdQ = JC_v dt + pdv.$$

EXAMPLES.

1. A raindrop falls to the ground from a height of 1272 metres; determine by how much its temperature is raised, assuming that it imparts no heat to the air or to the ground. *Ans.* 3° C.

2. Find how much heat is disengaged if a bullet weighing 50 grammes and having a velocity of 50 metres per second strikes a target, assuming g to be 9·8 metres per second.
Ans. An amount of heat sufficient to raise one gramme of water through 15° C.

3. In a perfect gas show that $C_p = C_v + \dfrac{R}{J}$.

4. Calculate the difference between the two specific heats of air, being given that a cubic metre of air, at a temperature of 0° C., and under a pressure of 760 mm. of mercury, whose density is 13·6, weighs 1·2932 kilogrammes. *Ans.* 0·069.

5. For any gas whose density referred to air is d, show that

$$C_p = C_v + \frac{0{\cdot}069}{d}.$$

6. Determine the quantity of heat which must be imparted to a gas to enable it to expand at a constant pressure p_1 from the volume v_1 to the volume v_2.

Ans. $Q = \frac{C_p}{R} p_1 (v_2 - v_1)$.

7. The volume and pressure of a gas whose mass is unity change from v_1 and p_1, to v_2 and p_2; determine the amount of heat which must be imparted to the gas in order that its temperature should remain constant.

If T_1 be the constant temperature, $JQ = RT_1 \log \frac{v_2}{v_1}$. The transformation of the gas in this case is said to be *isothermal.*

8. If a gas expand or contract without gain or loss of heat, show that the initial temperature and volume, T_1 and v_1, are connected with the final temperature and volume, T_2 and v_2, by the equation $\frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{k-1}$, where k is the ratio of the specific heat at constant pressure to the specific heat at constant volume. The transformation of the gas in this case is said to be *adiabatic.*

9. In the preceding example determine the equation connecting the initial and final pressures with the initial and final volumes.

Ans. $\frac{p_2}{p_1} = \left(\frac{v_1}{v_2}\right)^k$.

10. Determine the external work done by a gas in an isothermal transformation.

In this case $W = JQ = RT_1 \log \frac{v_2}{v_1} = p_1 v_1 \log \frac{v_2}{v_1}$.

11. Prove that the external work done by a gas in an adiabatic transformation is $JC_v(T_1 - T_2)$, where T_1 and T_2 are the initial and final temperatures.

12. If the decrease in the temperature of the air as its height above the surface of the earth increases were due merely to the fall of temperature resulting from the expansion caused by diminution of pressure, show that ΔT, the excess of the temperature at the earth's surface above the temperature at any height z, would be given by the equation

$$\Delta T = \frac{k-1}{k} \frac{273z}{h_0},$$

where h_0 is the height of a homogeneous atmosphere at 0° C.

If p be the pressure, and ρ the density of the air at the height z, we have, from the fundamental equation of hydrostatics, $dp = -g\rho\, dz$; but $\rho = \frac{1}{v} = \frac{p}{RT}$, and since the expansion is adiabatic, $\frac{dT}{T} = \frac{k-1}{k} \frac{dp}{p}$. Eliminating dp, we have $dT = -\frac{k-1}{k} \frac{g}{R} dz$, from which, since $gh_0 = p_0 v_0$, we obtain the equation given above.

13. If T be the absolute temperature of a gas, and $\mathfrak{T}'$ the portion of the energy of its unit mass which is due to the velocities of translation of its molecules, show that $\mathfrak{T}' = \frac{3}{2} RT$.

Since $pv = \frac{2}{3}\mathfrak{T}'$ (Ex. 25, p. 413), this result follows from (4) Art. 292.

14. Determine the mean velocity of translation of a molecule of air which is at a temperature of 0° C.

Here T is 273, and the mean velocity required is 485 metres per second, nearly.

15. Show that the mean velocities of translation of the molecules of different gases when at the same temperature are inversely proportional to the square roots of the densities of the gases.

16. Determine the relation between the total kinetic energy $\mathfrak{T}$ of a gas and that portion $\mathfrak{T}'$ of the kinetic energy which is due to the velocities of translation of the molecules.

The total energy U of the unit mass of a gas is composed of the kinetic energy $\mathfrak{T}$, and of the potential energy V, which again is the sum of two parts, V_1 resulting from the mutual action of the molecules, and V_2 depending on the constitution of the individual molecules. V_2 may be considered constant so long as the chemical constitution of the gas remains unchanged, and V_1 may be assumed to be zero, since U is a function of the temperature alone, and V_1, if it existed, would depend on the mutual distances of the molecules, and therefore on the volume. Hence $U = \mathfrak{T} + V_2$.

Again, $\dfrac{dU}{dT} = JC_v =$ constant, whence $U = JC_v T + C'$, or $\mathfrak{T} + V_2 = JC_v T + C'$.

Let $\mathfrak{T} = \beta\mathfrak{T}'$, then $\mathfrak{T} = \frac{3}{2}\beta RT$, and $\frac{3}{2}\beta RT + V_2 = JC_v T + C'$. Hence, as V_2 is constant, β must be of the form $\gamma + \dfrac{\gamma}{T}$, where γ and γ are constants; and $\mathfrak{T}$ must be the sum of two parts—one proportional to the temperature, the other constant. The existence of the latter part seems in the highest degree improbable; we may, therefore, conclude that β is constant. To determine its value we have $\frac{3}{2}\beta R = JC_v$, whence (Ex. 3) $\beta = \frac{2}{3}\dfrac{1}{k-1}$, where $k = \dfrac{C_p}{C_v}$. Now k is found to be almost the same for all gases, and to be equal to 1·408; hence β is approximately the same for all gases and is equal to 1·634.

17. Two masses of different gases have equal volumes at the same pressure and temperature; show that for all equal temperatures they have equal kinetic energies.

293. **Reversibility and Cyclical Processes.**—When a body experiences transformations such that the inverse changes can take place in precisely the same circumstances, the transformation is said to be *reversible*. In order that this should be the case, any source from which the body derives heat, or to which the body imparts heat, must, at the time at which the heat is transferred, be of the same temperature as the body; and also the external pressure on the body at any time

must be equal to the pressure corresponding to the state of the body at the time.

A cyclical process is a transformation at the end of which the body returns to the same state as that in which it was at the beginning.

294. **Indicator Diagram.**—The state of a body is, as we have seen, a function of two independent variables. If those selected be the volume of the unit of mass and the pressure on the unit of area, the state of the body at any time is indicated by the position of a point whose coordinates referred to two rectangular axes are proportional to the volume and pressure.

In the case of a body undergoing a transformation according to a fixed law, the set of points indicating its successive states form a curve. In a reversible transformation, if no heat be lost or gained by the body during the transformation, this curve is called an *adiabatic* or *isentropic* curve. If the temperature remain constant, the curve is called *isothermal.* The area comprised between the curve, its extreme ordinates, and the axis of abscissas, represents the work done by the body during the transformation.

295. **Fundamental Principles of Thermodynamics.**—The science of Thermodynamics is founded on two fundamental Principles. Of these, the first finds its mathematical expression in Equation (1), and involves two statements, viz., that *In every natural process the total energy is invariable;* and that *Heat is a form of energy, a definite amount of heat being equivalent to a definite amount of work.*

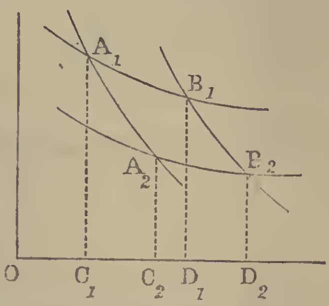
The second fundamental Principle was first stated by Clausius, as follows:—*It is impossible for a self-acting machine, unaided by external agency, to convey heat from one body to another at a higher temperature.*

By Thomson the same Principle is stated somewhat differently in the following manner:—*It is impossible by means of inanimate material agency to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects;* and by Clerk Maxwell in another form, thus:—*It is impossible, by the unaided action of natural processes, to transform any part of the heat of a body into me-*

chanical work, except by allowing heat to pass from that body into another at a lower temperature.

This Principle merely expresses the teaching of experience in reference to the connexion between temperature and the transference of heat.

296. **Carnot's Cycle.**—Let us suppose a body subject to a reversible cyclical process indicated by an isothermal A_1B_1, an adiabatic B_1B_2, another isothermal B_2A_2, at a lower temperature than the former, and another adiabatic A_2A_1. Whilst expanding from the volume represented by OC_1 to that represented by OD_1 the body is kept at a constant temperature t_1, receiving from a source K_1 of heat at that temperature a quantity of heat Q_1. Whilst expanding from OD_1 to OD_2, the temperature falls from t_1 to t_2, no heat being lost or gained. The body is now compressed from OD_2 to OC_2, and the heat Q_2 thereby developed is imparted to a reservoir K_2 at the temperature t_2. Finally, the body is compressed from OC_2 to OC_1, and the temperature thereby rises from t_2 to t_1, no heat being lost or gained. In this process the volume at which the adiabatic compression commences is selected in such a way that when the volume returns to its initial value the temperature returns likewise to its initial value. In the whole process Q_1 units of heat are imparted to the body, and Q_2 units of heat are taken from it; and as the body returns to its original state, $Q_1 - Q_2$ units of heat have been transformed into the work which is indicated by the area $A_1B_1B_2A_2$. In referring to this process K_1 and K_2 are frequently termed the *source* and the *condenser*.



We can now show that, if Q_1, t_1, and t_2 be supposed invariable, $Q_1 - Q_2$ must be the same for all bodies.

Suppose $Q_1 - Q_2$ for one body L were greater than for another body M, its value for L being denoted by Q, and for M by Q'. Employ the cyclical process for the body L to work that for the body M in reverse order. Then the source of heat at t_1 remains as before, whilst the source of heat at t_2 has received Q_2 and given out Q_2' units of heat. Moreover, an amount of work represented by $Q - Q'$ or $Q_2' - Q_2$ has been accomplished. The result of the whole process is that work has been done by means of heat obtained from the coldest body in the system. As this result is opposed to the second fundamental Principle (Art. 295), we conclude that $Q_1 - Q_2$ is the same for all bodies, and is therefore simply a function of Q_1, t_1, and t_2.

From what has been now proved it follows that if we suppose the curve $A_1 B_1$ divided into n parts, for each of which Q_1 is the same, and adiabatics drawn through the points of section, the corresponding values of Q are equal. Hence it is easy to see that Q becomes nQ if Q_1 become nQ_1, and therefore that $\frac{Q}{Q_1}$ must be independent of Q_1: accordingly, we have

$$\frac{Q}{Q_1} = f(t_1, t_2). \tag{5}$$

297. **Determination of Carnot's Function.**—In order to determine the function f, we have merely to select a body for which the isothermal and adiabatic curves are known. Let us then select a perfect gas.

In this case

$$JQ_1 = RT_1 \log \frac{v_{B_1}}{v_{A_1}}, \quad JQ_2 = RT_2 \log \frac{v_{B_2}}{v_{A_2}} \text{ (Ex. 7, p. 430).}$$

Again, as the points A_1 and A_2 lie on the same adiabatic, we have

$$\left(\frac{v_{A_1}}{v_{A_2}}\right)^{k-1} = \frac{T_2}{T_1}, \text{ and likewise } \left(\frac{v_{B_1}}{v_{B_2}}\right)^{k-1} = \frac{T_2}{T_1} \text{ (Ex. 8, p. 430);}$$

therefore $$\frac{v_{B_1}}{v_{A_1}} = \frac{v_{B_2}}{v_{A_2}}, \text{ and } \frac{Q_1}{Q_2} = \frac{T_1}{T_2}.$$

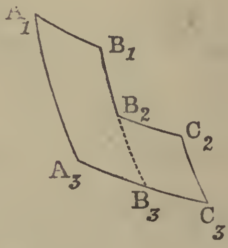
Hence, whatever be the body employed, we obtain the equation

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}. \tag{6}$$

298. **Extension of Carnot's Cycle.**—If heat imparted to a body be regarded as positive, and heat given out by the body as negative, (6) may be written

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} = 0. \tag{7}$$

If we now suppose a reversible cyclical process represented by any number of isothermals and adiabatics, each isothermal being followed by an adiabatic, and if Q be the whole quantity of heat imparted to the body at the temperature T, we have the equation $\Sigma \frac{Q}{T} = 0$.



In order to prove this, let us first suppose a cycle in which there are three isothermals, A_1B_1, B_2C_2, and A_3C_3, corresponding to the temperatures T_1, T_2, and T_3. Produce the adiabatic B_1B_2 to B_3, then $Q_3 = q_3 + q_3'$, where q_3 corresponds to B_3A_3, and q_3' to C_3B_3.

Now by (7),

$$\frac{Q_1}{T_1} + \frac{q_3}{T_3} = 0, \text{ and } \frac{Q_2}{T_2} + \frac{q_3'}{T_3} = 0;$$

from which, by addition, we have

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} + \frac{Q_3}{T_3} = 0.$$

This result may be extended in a similar manner to a cycle containing four isothermals, and so on. Hence, in general,

$$\Sigma \frac{Q}{T} = 0. \tag{8}$$

Again (8) holds good for every reversible cyclical process, whatever be the nature of the curves by which it is represented.

This appears from the consideration that two infinitely near points A and B on any curve can be connected by the element of an isothermal followed by that of an adiabatic, and that the area bounded by these elements, the ordinates of A and B, and the axis of abscissas, differs only by an infinitely small quantity of the second order from the area of which the arc AB is the boundary.

For every reversible cyclical process, however effected, we have, then, the equation

$$\int \frac{dQ}{T} = 0. \tag{9}$$

299. **Entropy.**—If a body pass from any one state to any other, we may suppose the change of state effected by means of a reversible transformation; and whatever this process be, $\int \frac{dQ}{T}$ between the limits corresponding to the two states must have the same value, since the cycle may be completed by a definite invariable transformation. Hence $\int \frac{dQ}{T}$ depends only on the state of the body, and is independent of the mode (supposed reversible) by which the body is brought into this state.

If we put $\int \frac{dQ}{T} = \phi$, the quantity ϕ is called by Clausius the *Entropy* of the body.

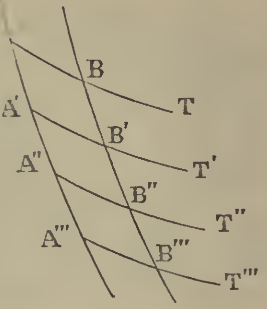
The second fundamental Principle of Thermodynamics leads therefore to the result, that the entropy ϕ is a function of the two independent variables on which the state of the body depends, and therefore that in all reversible transformations $\frac{dQ}{T}$ is a perfect differential $d\phi$, or that

$$dQ = Td\phi. \tag{10}$$

In theoretical applications of the equation of entropy, Q and ϕ are supposed to be expressed in *mechanical* units.

300. **Absolute Scale of Temperature.**—The result obtained in Art. 297 may be arrived at by a different method independent of the properties of any particular substance. We have seen, Art. 296, that if H be the heat drawn from

the source, and W the heat converted into work in Carnot's cycle, $\frac{W}{H}$ is a function of the extreme temperatures only, and is independent of the substance employed. In order, then, to construct a scale of temperature independent of any particular body we may proceed as follows:—



Draw the isothermal AB of a body chosen at random, corresponding to any arbitrary temperature which may be indicated by T, and draw the adiabatics AA' and BB' corresponding to the condition of the body before and after a certain arbitrary amount of heat H has been imparted to it.

Draw another isothermal at a temperature T' less than T, so that the area $ABB'A'$ may be of given magnitude or correspond to a given amount of heat w. Now draw a series of isothermals T'', T''', &c., at intervals such that

$$ABB'A' = A'B'B''A'' = A''B''B'''A''' = \&c.;$$

then if $T - T'$ be the unit of temperature, $T - T''$ is two units, $T - T'''$ three units, &c.

Since T, H, and w are fixed quantities, and W corresponding to $T^{(n)}$ is nw, Equation (5) shows that two bodies are at the same temperature if each indicates in the manner described n degrees of temperature below T. This method of estimating temperature is, therefore, independent of the body employed.

If T' be *any* temperature lower than T estimated in this manner, and W' the heat converted into work in the corresponding cyclical process, since

$$\frac{w}{H} = F(T), \text{ and } W' = (T - T')w,$$

we have $W' = H(T - T')F(T)$. If $F(T)$ be denoted by C, this equation may be written $W' = HC(T - T')$. In like manner, $W'' = HC(T - T'')$. If we now suppose a cyclical process between the temperatures T' and T'', indicated by the points A', B', B'', A'', the heat converted into work is $W'' - W'$; also H', the heat drawn from the source at T', is equal to that given to the condenser in the process in which T and T' are the extreme temperatures, hence

$$H - H' = W' = HC(T - T').$$

301. **Efficiency of a Heat Engine.**—A system working in the manner required by Carnot's cycle may be termed a reversible heat engine, and the ratio of the heat converted into work to the heat drawn from the source is called the *efficiency of the engine.*

It appears by the reasoning of Art. 296 that the *extreme temperatures being given, the efficiency of a non-reversible engine cannot exceed that of a reversible, and that the efficiency of all reversible engines is the same.*

302. **Absolute Zero.**—It follows from Art. 300 that the efficiency of a reversible engine working between the temperatures T' and T'' is

$$\frac{HC(T' - T'')}{H - HC(T - T')} \text{ or } \frac{T' - T''}{T' - \left(T - \frac{1}{C}\right)}.$$

As T'' decreases, the efficiency increases, but the limit which it can never exceed is unity, since the mechanical work done by an engine can never exceed the equivalent of the heat drawn from the source. Hence, if we make the efficiency unity, we obtain for T'' the smallest possible value, which is $T - \frac{1}{C}$. This temperature T'', since it is the lowest which can be attained by any body, must be the absolute zero. Hence

$$T - \frac{1}{C} = 0, \text{ and } C = \frac{1}{T}.$$

The expression for the efficiency of a reversible engine working between any two temperatures T' and T'' becomes then $\frac{T' - T''}{T'}$. Carnot's function C has thus been determined independently of the properties of any particular substance.

Again, this mode of determining Carnot's function shows that the existence of an absolute zero of temperature, suggested and rendered probable by the known properties of what are called permanent gases, follows necessarily from the two fundamental Principles of Thermodynamics.

The experiments of Joule and Thomson have shown that the absolute zero is 273·7 below zero on the Centigrade scale, or 460·66 below zero on the Fahrenheit. This is very nearly the same result as that of Article 292.

303. **Energy and Entropy.**—For every reversible transformation in which the external work done by the body is due to its own expansion we have, if Q be expressed in work units, the two equations

$$\left.\begin{aligned} dQ &= dU + p\,dv \\ dQ &= Td\phi \end{aligned}\right\}. \qquad (11)$$

U and ϕ are functions of the independent variables on which the state of the body depends, and dU and $d\phi$ are therefore perfect differentials; Q depends not merely on the state of the body, but also on the mode in which it has been brought into that state; hence dQ is not a perfect differential. The limits of the quantities $\frac{\Delta Q}{\Delta v}$, $\frac{\Delta Q}{\Delta p}$, &c., are expressible in terms of the independent variables and the differential coefficients of U and v. They are therefore functions of the two independent variables which determine the state of the body, but are not differential coefficients. They may be written $\frac{\delta Q}{\delta v}$, &c.

If we eliminate dQ between the two equations (11), we have

$$dU = Td\phi - pdv. \qquad (12)$$

From this equation numerous results can be deduced, the most important of which will be found among the examples.

304. **Non-reversible Transformations.**—In the case of a non-reversible transformation we cannot assume the truth of equations (11). In fact, for such a transformation, even though the external work done by the body be due to its expansion against external pressure, this pressure need not be equal in magnitude to that belonging to the state of the body, nor is dQ in such a transformation necessarily equal to $Td\phi$.

In this case we must proceed as follows:—Let H be the whole amount of heat actually imparted to the body in the non-reversible process, W the external work done, and U_0 and U the initial and final energies of the body; then

$$H = U - U_0 + W. \tag{13}$$

Let us now imagine a reversible transformation capable of bringing the body from its initial to its final state, but otherwise perfectly arbitrary. Since this hypothetical transformation is reversible, we can make use of equations (11) and (12), and any results therefrom deducible to assist in determining $U - U_0$. The expression thus obtained may be substituted in (13).

305. **Change of State.**—There are three states or conditions in one of which matter is usually found; and which are termed the *solid*, the *liquid*, and the *gaseous*.

A solid body is one which strongly resists any forces tending to alter the relative positions of its adjacent molecules, and, so long as its structure is unbroken, admits of only slight changes in these positions.

A liquid offers scarcely any resistance to a change in the mutual position of its molecules, provided this change does not diminish the distance between those which are adjacent. In other words, it is indifferent to change of shape or separation of its molecules from each other, but strongly resists compression.

A gas, like a liquid, is indifferent to change of shape, but yields to compression with comparative facility, and tends to increase its volume without limit. To prevent its escape, it must therefore be restrained by an external envelope.

It is almost certain that every substance in nature is capable of existing in any one of the three states, and passes at a certain pressure and temperature from one of these states into another.

Thus, in general, there is a certain pressure and temperature at which, if heat be imparted to a solid body and the pressure be maintained constant, the temperature does not rise, but the body gradually passes into the liquid state. The amount of heat required to bring about this change for the unit of mass is called the *latent heat of liquidity.* If the volume of the liquid exceed that of the solid, the latent heat of liquidity is spent partly in altering the internal energy and partly in doing external work. If (as is the case with water) the volume of the liquid be less than that of the solid, the internal energy of the liquid exceeds that of the solid by an amount greater than the equivalent of the latent heat of liquidity.

The vapour of a liquid may be considered a gas. If the temperature be sufficiently high, and the pressure sufficiently low, a vapour obeys the same laws as the gases which are called permanent, and approaches closely to the condition of a perfect gas (Art. 292).

For each vapour there is, corresponding to any given temperature T_1, a certain pressure p_1, such that at higher pressures the vapour begins to liquify; and, conversely, corresponding to any given pressure p_1 there is a temperature T_1, such that at lower temperatures the same result takes place. The pressure p_1 is called the maximum pressure of the vapour for the temperature T_1, and T_1 is called the boiling point of the liquid for the pressure p_1.

In fact, if heat be imparted to the liquid under the constant pressure p_1, the temperature of the liquid will rise until it reaches T_1: after this the temperature will remain stationary, but the liquid will be transformed into vapour, and the heat required by the unit mass for this transformation is called the *latent heat of vaporization.* When a vapour is at its maximum pressure, and therefore beginning to liquify, it is said to be *saturated.*

A liquid exposed to the air evaporates more or less at all temperatures. It is known that if two gases be enclosed in the same envelope, each, after some time, is diffused through the whole volume of the envelope as if the other were absent. Hence we might anticipate the behaviour of a liquid exposed to the atmosphere, and expect that the air would not act like an impervious envelope exercising a constant pressure, but would merely retard the formation of vapour by diminishing its rate of diffusion.

The statements of Article 290 require some modification when applied to a body while changing its state. So long as the state remains unaltered, U, ϕ, and v are functions of p and T; but when the body begins to change its state, U, ϕ, and v vary, even though p and T remain constant. Whilst the body is changing its state, if p and T be constant, U, ϕ, and v are functions of a single variable. If the change of state go on continuously, and p at the same time vary, then T must also vary, and be at each instant a function of p. In this case, U and ϕ are functions of two independent variables, which may be v and p, or v and T, but cannot be p and T.

The experiments of Andrews, and the investigations of Thomson, have thrown much light on the phenomena of change of state, and enable us to explain their seeming anomalies. An account of these researches would, however, be outside the limits of the present work.

306. **Elasticity and Expansion.**—The *elasticity* of a substance may be defined as the limit of the ratio of an increase of pressure to the compression which it produces, the compression being the ratio of the diminution of volume to the original volume.

The state of a substance being determined by two independent variables, except some connexion between their variations be assigned, the elasticity is indeterminate. The two elasticities usually considered are the elasticity at constant temperature E_T, and the elasticity at constant entropy E_ϕ. The former obviously belongs to an isothermal, and the latter to an adiabatic, transformation.

From the definitions of E_T and E_ϕ we have

$$E_T = -v\left(\frac{dp}{dv}\right)_T, \quad E_\phi = -v\left(\frac{dp}{dv}\right)_\phi.$$

When a body is heated it usually expands. The *expansion* is the ratio of the increase of volume to the original volume, and the *expansibility* is the limit of the ratio of the expansion to the increase of temperature, the pressure remaining constant. If e denote the expansibility, we have

$$e = \frac{1}{v}\left(\frac{dv}{dT}\right).$$

If the expansion of a body take place without change of temperature, the limit of the ratio of the heat required for the expansion to the increase of volume is called the *latent heat of expansion*, and may be denoted by l; hence

$$l = \frac{1}{J}\left\{\left(\frac{dU}{dv}\right)_T + p\right\}.$$

EXAMPLES.

1. Determine the form of the isothermal curves for a perfect gas.

Ans. Equilateral hyperbolas.

2. In the same case determine the equation of an adiabatic.

Ans. $pv^k =$ constant.

3. The volume and pressure of a gas being given, determine its entropy.

Ans. $\phi - \phi_0 = J\left\{C_v \log\frac{p}{p_0} + C_p \log\frac{v}{v_0}\right\}.$

4. Prove that the following equations hold good in any reversible transformation in which the external work done by a body is due to its expansion against the pressure on its surface:

$$\left(\frac{dT}{dv}\right)_\phi = -\left(\frac{dp}{d\phi}\right)_v, \quad \left(\frac{dT}{dp}\right)_\phi = \left(\frac{dv}{d\phi}\right)_p, \quad \left(\frac{d\phi}{dp}\right)_T = -\left(\frac{dv}{dT}\right)_p,$$

$$\left(\frac{d\phi}{dv}\right)_T = \left(\frac{dp}{dT}\right)_v, \quad \left(\frac{dT}{dp}\right)_v\left(\frac{d\phi}{dv}\right)_p - \left(\frac{dT}{dv}\right)_p\left(\frac{d\phi}{dp}\right)_v = 1.$$

These equations follow from (12), Art. 303, by choosing successively for independent variables ϕ, v; ϕ, p; T, p; T, v; and v, p; and expressing in each case the condition that dU should be a perfect differential. Briot remarks that from the first of these equations the three succeeding can be obtained by interchanging p and v or T and ϕ, the sign of the right-hand member of the equation being altered after each interchange.

5. Show that

$$JC_v = T\left(\frac{d\phi}{dT}\right)_v, \quad JC_p = T\left(\frac{d\phi}{dT}\right)_p, \quad J(C_p - C_v) = -T\frac{\left(\frac{dp}{dT}\right)_v^2}{\left(\frac{dp}{dv}\right)_T}.$$

The first two of these equations follow at once from (12), p. 439. To prove the last, we have from the two former,

$$J(C_p - C_v) = T\left\{\left(\frac{d\phi}{dT}\right)_p - \left(\frac{d\phi}{dT}\right)_v\right\};$$

but $$\left(\frac{d\phi}{dT}\right)_p = \left(\frac{d\phi}{dT}\right)_v + \left(\frac{d\phi}{dv}\right)_T\left(\frac{dv}{dT}\right)_p;$$

and by Ex. 4,

$$\left(\frac{d\phi}{dv}\right)_T = \left(\frac{dp}{dT}\right)_v; \text{ hence } J(C_p - C_v) = T\left(\frac{dp}{dT}\right)_v\left(\frac{dv}{dT}\right)_p.$$

Again, $dp = \left(\frac{dp}{dT}\right)_v dT + \left(\frac{dp}{dv}\right)_T dv$, from which, by making $dp = 0$, we get $\left(\frac{dv}{dT}\right)_p$, and substituting in the expression for $J(C_p - C_v)$ already given, we obtain the result required.

6. Prove that $\frac{C_p}{C_v} = \frac{E_\phi}{E_T}$.

$$\frac{C_p}{C_v} = \frac{\left(\frac{d\phi}{dT}\right)_p}{\left(\frac{d\phi}{dT}\right)_v} = \frac{\left(\frac{d\phi}{dv}\right)_p\left(\frac{dv}{dT}\right)_p}{\left(\frac{d\phi}{dp}\right)_v\left(\frac{dp}{dT}\right)_v}; \quad \text{but} \quad d\phi = \left(\frac{d\phi}{dv}\right)_p dv + \left(\frac{d\phi}{dp}\right)_v dp,$$

and therefore $\frac{\left(\frac{d\phi}{dv}\right)_p}{\left(\frac{d\phi}{dp}\right)_v} = -\left(\frac{dp}{dv}\right)_\phi$. In like manner $\frac{\left(\frac{dT}{dp}\right)_v}{\left(\frac{dT}{dv}\right)_p} = -\left(\frac{dv}{dp}\right)_T$;

hence $$\frac{C_p}{C_v} = \frac{\left(\frac{dp}{dv}\right)_\phi}{\left(\frac{dp}{dv}\right)_T} = \frac{E_\phi}{E_T} \text{ (Art. 306).}$$

7. If λ be the latent heat of change of state expressed in work units, and v the increase in the volume of the unit mass of the substance after passing from one state to the other, prove that

$$\lambda = vT\frac{dp}{dT}.$$

Let s and σ be the volumes of the unit mass of the substance before and after the change of state, and μ the fraction of the unit mass which has undergone the change at any instant during the transformation, then $v = \sigma - s$, and $v = \mu\sigma + (1 - \mu)s = s + v\mu$. Again, if Q be the quantity of heat required to transform μ times the unit mass from one state to the other, the pressure and temperature remaining constant, $Q = \lambda\mu$, and therefore

$$\lambda = \left(\frac{\delta Q}{\delta\mu}\right)_T = T\left(\frac{d\phi}{d\mu}\right)_T = vT\left(\frac{d\phi}{dv}\right)_T = vT\frac{dp}{dT}. \quad \text{(Art. 305, and Ex. 4.)}$$

The student will observe that p is here a function of T alone, and that if L be the latent heat expressed in heat units, $\lambda = JL$.

8. The density of ice being 0·92, and the latent heat of water 79·25, find the lowering of the temperature of freezing caused by an additional pressure of one atmosphere. *Ans.* 0·0073° C.

9. If c_s and c_w be the specific heats expressed in work units of saturated steam and of boiling water at the same pressure, show that

$$c_s - c_w = \frac{d\lambda}{dT} - \frac{\lambda}{T}.$$

It is here supposed that the variations of T and p are so related (Art. 291) that as T changes the steam remains saturated, and the water remains boiling (Art. 305). Hence, if we suppose μ to remain constant (Ex. 7), we have

$$c_s\mu + c_w(1 - \mu) = \left(\frac{\delta Q}{\delta T}\right)_\mu = T\left(\frac{d\phi}{dT}\right)_\mu;$$

and therefore

$$c_s - c_w = T\frac{d^2\phi}{dTd\mu} = T\frac{d}{dT}\left(\frac{d\phi}{d\mu}\right)_T;$$

but

$$\left(\frac{d\phi}{d\mu}\right)_T = \frac{\lambda}{T}, \quad \text{(Ex. 7)};$$

hence, substituting and performing the differentiation, we have the result required. It may be observed that c_w does not sensibly differ from the specific heat of water at constant pressure.

10. Investigate a numerical formula for the specific heat of saturated steam at any given temperature.

According to Regnault, the whole quantity of heat required to raise the temperature of the unit mass of water from 0° to t° C. and evaporate it at that temperature is $606{\cdot}5 + 0{\cdot}305\,t$. Hence we have the empirical formula

$$L + \int_0^t C_w\,dt = 606{\cdot}5 + 0{\cdot}305\,t,$$

where C_w is the specific heat of boiling water expressed in heat units. Differentiating, we have $\dfrac{dL}{dt} + C_w = 0{\cdot}305$. For high temperatures Regnault's empirical formula may be replaced by the simpler formula of Clausius, viz., $L = 607 - 0{\cdot}708\,t$. If we express $\frac{L}{T}$ by means of the latter we have (Ex. 9),

$$C_s = 0{\cdot}305 - \frac{607 - 0{\cdot}708\,t}{273 + t} = 1{\cdot}013 - \frac{800{\cdot}3}{273 + t}.$$

For temperatures near 0° C. we may take $C_w = 1$; we thus get

$$C_s = 1 - \frac{796{\cdot}2}{273 + t}.$$

From the expressions obtained for C_s we may conclude that, except the temperature be enormously high, the specific heat of saturated steam is negative. Hence it follows, that if saturated steam be compressed, the temperature after compression will be higher than that corresponding to saturation at the new pressure; or, in other words, saturated steam suffers no condensation, but becomes super-heated by adiabatic compression. Conversely, if saturated steam be contained in a vessel impervious to heat, a diminution of pressure will cause partial condensation. These results were first obtained theoretically by Clausius and Rankine, who, independently, arrived at them almost simultaneously. They have since been confirmed experimentally by Hirn. It seems not unlikely that the connexion between rain and a change of atmospheric pressure depends partly on the property of steam mentioned above.

11. If U_1 be the energy of the unit mass of saturated steam at T_1, and U_0 that of the unit mass of boiling water at T_0, prove that

$$U_1 = U_0 + \int_{T_0}^{T_1} \left(c_w - p\,\frac{ds}{dT}\right) dT + \lambda_1 - p_1 v_1.$$

Let us suppose that the unit mass is brought from the state of boiling water at T_0 and p_0 to that of saturated steam at T_1 and p_1, and that this transformation is effected by first bringing the water without evaporation, but continually boiling, from p_0, T_0, to p_1, T_1; and then evaporating at T_1, p_1.

Take as variables T and μ (Ex. 7); then, since $dU = dQ - p\,dv$, we have

$$dU = \left\{ \left(\frac{\delta Q}{\delta T}\right)_\mu - p\left(\frac{dv}{dT}\right)_\mu \right\} dT + \left\{ \left(\frac{\delta Q}{\delta \mu}\right)_T - p\left(\frac{dv}{d\mu}\right)_T \right\} d\mu.$$

Again, on the above hypothesis, when T varies μ is zero, and when μ varies T is constant. In general (Ex. 9),

$$\left(\frac{\delta Q}{\delta T}\right)_\mu = (c_s - c_w)\,\mu + c_w,$$

and therefore when $\mu = 0$, $\left(\frac{\delta Q}{\delta T}\right)_\mu = c_w$.

Again,

$$\left(\frac{dv}{dT}\right)_\mu = \frac{ds}{dT}, \quad \left(\frac{\delta Q}{\delta \mu}\right)_T = \lambda, \quad \left(\frac{dv}{d\mu}\right)_T = v, \text{ (Ex. 7).}$$

Hence, substituting, we have

$$U_1 = U_0 + \int_{T_0}^{T_1} \left(c_w - p\,\frac{ds}{dT}\right) dT + (\lambda_1 - p_1 v_1) \int_0^1 d\mu.$$

12. Assuming that the square of the velocity of the propagation of sound is proportional to the elasticity of the medium divided by its density, show that in a gas the velocity of sound varies as $\sqrt{kRT}$.

Since the compression of the air during the passage of a wave of sound is very sudden, the compression may be regarded as adiabatic. Hence the velocity of sound varies as $\sqrt{E_\phi v}$, but $E_\phi = kE_T$ (Ex. 6), and $E_T = p$, therefore, &c.

By means of the results obtained in this example and in Ex. 3, p. 429, if the velocity of sound be determined by experiment, C_p and C_v can be calculated. Conversely, if C_p be known by experiment, C_v can be found from the velocity of sound, and hence the value of J can be determined.

13. Show that bodies which expand by heating are heated by compression; those which contract by heating are cooled by compression; and, if the temperature be maintained constant, determine the rate at which heat is given out or absorbed according as the pressure is increased.

If Q be the heat required to keep the temperature constant, the rate of absorption is $\left(\frac{\delta Q}{\delta p}\right)_T$; but

$$\left(\frac{\delta Q}{\delta p}\right)_T = T\left(\frac{d\phi}{dp}\right)_T = -\;T\left(\frac{dv}{dT}\right)_p = -\,veT \text{ (Ex. 4, and Art. 306).}$$

Hence δQ is negative if e be positive, and conversely.

14. Prove that in water not far from its maximum density the rise of temperature produced by an increase of pressure is given by the formula

$$\Delta t = \frac{(t + 273)(t - 4)}{2950000}\,\Delta p,$$

where t is expressed in degrees centigrade, and p in atmospheres.

If v_0 be the volume of the unit mass of water at 4°, when the density is a maximum, the empirical formula $v = v_0\left(1 + \frac{(t-4)^2}{144000}\right)$ represents, according to Kopp and Tait, the results of numerous experiments. From this formula we have approximately $e = \frac{t-4}{72000}$.

Hence, assuming the pressure of the atmosphere to be 1033 grammes on the square centimetre, we obtain the required result.

15. A gas at p_1 and v_1 is allowed to expand into a perfectly empty vessel, whereby its pressure and volume become p_2, and v_2. No heat being imparted to the gas or taken from it, determine its change of temperature.

Since no external work is done, and no heat lost or gained, the energy must remain constant, and therefore the temperature (Art. 292). The assumption that the energy of a gas is a function of the temperature alone, is indeed a result from the fact ascertained by experiment, that the temperature is constant under the circumstances here supposed.

16. In the preceding example determine the change of entropy.

We have (Ex. 3),

$$\phi_2 - \phi_1 = J\left(C_v \log \frac{p_2}{p_1} + C_p \log \frac{v_2}{v_1}\right) = J(C_p - C_v) \log \frac{v_2}{v_1} = R \log \frac{v_2}{v_1}.$$

Since $v_2 > v_1$, the entropy is increased by the supposed transformation, which, it should be observed, is non-reversible, and therefore not adiabatic, though no heat is lost or gained.

17. Gas is made to pass uniformly through a tube in which a porous plug such as cotton-wool is placed. No heat is permitted to leave the gas or enter it from any external source; determine the connexion between the variations of pressure and temperature caused by the plug.

Since the density of the gas at any particular cross section of the tube does not vary during the experiment, equal masses of gas pass through each section in the same time, or the velocity of the unit of mass is constant. Again, any energy which is lost by friction is restored as heat. We are therefore entitled to assume that any change in the energy of the gas as it passes through different parts of the tube is due to the work done on it or to the work which it does.

Suppose two cross sections A and B of the tube, one on each side of the plug, the pressures at which are p_1 and p_2. As a small quantity dm of gas passes A the pressure driving it forward does work on it whose amount is $p_1 v_1\, dm$. At the same time dm does work on the next layer of gas which is equal to the work done on dm when passing the section consecutive to A. Thus, in going from A to B the work done by dm and the work done on dm compensate each other, with the exception of $p_1 v_1\, dm$ done on dm, and $p_2 v_2\, dm$ done by dm. In other words, in the passage from A to B the whole external work done by dm is $(p_2 v_2 - p_1 v_1)\, dm$, and therefore, since no heat is lost or gained, we have

$$U_2 - U_1 + p_2 v_2 - p_1 v_1 = 0.$$

Now $U = \int T d\phi - \int p\, dv = \int T d\phi - pv + \int v\, dp$, or $U + pv = \int T d\phi + \int v\, dp$.

Hence the hypothetical reversible transformation (Art. 304) must be such that

$$\int\left\{T\left(\frac{d\phi}{dT}\right)_p dT + \left(T\left(\frac{d\phi}{dp}\right)_T + v\right) dp\right\} = 0.$$

Substituting c_p for $T\left(\frac{d\phi}{dT}\right)_p$ (Ex. 5), and $-\left(\frac{dv}{dT}\right)_p$ for $\left(\frac{d\phi}{dp}\right)_T$ (Ex. 4),

we have
$$\int\left\{c_p\, dT + \left(v - T\left(\frac{dv}{dT}\right)_p\right) dp\right\} = 0.$$

Hence, if we can integrate the expression under the integral sign in this equation, the relation between $T_2 - T_1$ and $p_2 - p_1$ is determined. If the gas were theoretically perfect (Art. 292), we should have $T\left(\frac{dv}{dT}\right)_p = v$, and T_2 would be equal to T_1. This is found not to be the case. We may therefore conclude that no gas is theoretically perfect, and we cannot assume either that α is constant or that $T = t + \frac{1}{\alpha}$.

From the equation $vp = v_0 p_0 (1 + \alpha t)$, if we consider α as variable, we have

$$v - T\left(\frac{dv}{dt}\right)_p = \frac{p_0 v_0}{p}\left(1 + \alpha t - \alpha T - tT\frac{d\alpha}{dt}\right).$$

We may assume that $T - t$ differs from a constant by a very small quantity, and likewise that $\frac{d\alpha}{dt}$ is very small, and may integrate between T_1 and T_2 neglecting these quantities, since $T_1 - T_2$ is observed to be small. For similar reasons we may assume that $\left(\frac{dv}{dT}\right)_p = \left(\frac{dv}{dt}\right)_p$, and that c_p is constant. Hence, integrating, we have

$$-c_p(T_2 - T_1) = p_0 v_0 (1 + \alpha t - \alpha T)(\log p_2 - \log p_1),$$

and therefore

$$T = t + \frac{1}{\alpha} + \frac{c_p(T_1 - T_2)}{\alpha p_0 v_0 (\log p_1 - \log p_2)}.$$

From this equation the exact position on the centigrade scale of the absolute zero can be determined (Art. 302).

18. Saturated steam in a vessel containing no water is allowed to escape into the air; determine the quantity of heat which must be imparted to the unit of mass in order that it should remain saturated.

Let W be the external work done by the unit mass of the steam in escaping, T_1 and T_2 its initial and final temperatures; then H being the heat required, we have

$$JH = U_2 - U_1 + W.$$

Now $$W = p_2(\sigma_2 - \sigma_1) = p_2(v_2 - v_1) + p_2(s_2 - s_1), \quad \text{(Ex. 7)},$$

where σ and s are the volumes of the unit mass of steam and of water,

and $$U_2 - U_1 = \int_{T_1}^{T_2}\left(c_w - p\frac{ds}{dT}\right)dT + \lambda_2 - \lambda_1 - p_2 v_2 + p_1 v_1, \quad \text{(Ex. 11)}.$$

Hence $$JH = \int_{T_1}^{T_2}\left(c_w - p\frac{ds}{dT}\right)dT + \lambda_2 - \lambda_1 + v_1(p_1 - p_2) + p_2(s_2 - s_1).$$

Since $(s_2 - s_1)$ and $\frac{ds}{dT}$ are small, we may neglect them, and thus obtain

$$JH = \int_{T_1}^{T_2}\left(c_w + \frac{d\lambda}{dT}\right)dT + v_1(p_1 - p_2).$$

Now $$\frac{1}{J}\left(c_w + \frac{d\lambda}{dT}\right) = 0{\cdot}305, \quad \text{(Ex. 10)};$$

hence $$H = -\,0{\cdot}305\,(T_1 - T_2) + (p_1 - p_2)\,\frac{L_1}{T_1\left(\dfrac{dp}{dT}\right)_1}, \quad \text{(Ex. 7)}.$$

$\left(\dfrac{dp}{dT}\right)_1$ is the limit of the ratio of the change of the maximum pressure of the vapour to the corresponding change of temperature, and is hence easily found from a table of the temperatures of the boiling point at different pressures. H is in general positive; *i.e.*, if no heat be imparted to the expanding steam, some of it will condense.

19. In the preceding example, if the vessel from which the steam is escaping contain boiling water, determine the quantity of heat which must be imparted to the unit mass of steam in order that it should remain saturated.

In this case, as in Ex. 17, the external work W done by the unit mass of the steam is

$$p_2\sigma_2 - p_1\sigma_1, \text{ or } p_2v_2 - p_1v_1 + p_2s_2 - p_1s_1.$$

Hence, as $U_2 - U_1$ is the same as in the last example, if we neglect the comparatively small terms involving s_1 and s_2, we have approximately

$$JH = \int_{T_1}^{T_2}\left(c_w + \frac{d\lambda}{dT}\right)dT,$$

and therefore $$H = -\,0{\cdot}305\,(T_1 - T_2).$$

In this case, if no heat be abstracted, or imparted, the steam after it escapes is super-heated. If $T_1 - T_2$ be large, or the steam originally at high pressure, the super-heating is considerable and more than sufficient to vaporize any particles of water which the steam carries with it mechanically. Hence we can explain the known phenomenon that high pressure steam after escaping into the air is dry and does not scald, whereas by low pressure steam severe scalds may be inflicted.

307. **Available Energy.**—The work which can be accomplished by a quantity of heat Q_1 depends on the temperature of the source from which it is derived. If this temperature be T_1, and the lowest temperature which can be obtained T_0, the work which can be accomplished by means of Q_1 cannot exceed $(T_1 - T_0)\,\dfrac{Q_1}{T_1}$ (Art. 302), where Q_1 is expressed in work units.

If Q_1 pass from a source at T_1 to a source at T_2, the available energy is diminished by the quantity $\left(\dfrac{T_0}{T_2} - \dfrac{T_0}{T_1}\right)Q_1$.

If Q_1 leave a source at T_1, and Q_2 in consequence enter a source at T_2, the loss of available energy is

$$(T_1 - T_0)\,\frac{Q_1}{T_1} - (T_2 - T_0)\,\frac{Q_2}{T_2}, \text{ or } Q_1 - Q_2 - T_0\left(\frac{Q_1}{T_1} - \frac{Q_2}{T_2}\right).$$

308. **Dissipation of Energy.**—If the transference of heat from a source at T_1 to a source at T_2 take place through the medium of a reversible engine undergoing a cyclical process, $\frac{Q_1}{T_1} - \frac{Q_2}{T_2}$ is zero, and the loss of available energy is $Q_1 - Q_2$, which is the same as the work done. Thus the uncompensated loss of available energy is zero.

In the case of an engine undergoing a non-reversible cyclical process, $Q_1 - Q_2$ cannot be greater, and is usually less, than $(T_1 - T_2)\frac{Q_1}{T_1}$, (Art. 301), or $\frac{Q_1}{T_1} - \frac{Q_2}{T_2}$ has a negative value which may be denoted by $-N$. In this case the uncompensated loss of available energy is $T_0 N$.

By a method similar to that employed in Art. 298 this result can be extended to every non-reversible cyclical process. In this case, if Q be the heat which enters the engine at the temperature T, the quantity $\Sigma\frac{Q}{T}$ is negative, and the uncompensated loss of available energy is $-T_0\Sigma\frac{Q}{T}$.

To prove this, we have only to substitute for the actual process A a process B, in which the cycles corresponding to each pair of temperatures are completed by reversible transformations, each of which is accomplished first in one direction, then in the opposite. As these transformations are passed through in both directions, the value of $\Sigma\frac{Q}{T}$ and of the uncompensated loss of available energy is the same for A as for B; but $\Sigma\frac{Q}{T}$ for B is the sum of the values of $\Sigma\frac{Q}{T}$ corresponding to the small cycles, since the remaining part of B forms one reversible cycle. Hence we obtain the required results.

The uncompensated loss of available energy is called the *Dissipation of Energy*.

From the present and preceding Articles it appears that this dissipation takes place whenever heat passes without the performance of work from a body at a higher to a body at a lower temperature, and also, in general, in non-reversible cyclical processes. A strictly reversible process cannot be realized in nature, since the absence of friction and the perfect equality of internal and external pressures and temperatures cannot be attained. Hence we may conclude, that in all natural processes there is an incessant dissipation of energy. It is to be observed that the theory of dissipation depends on the assumption of a certain temperature as the lowest which is available. If the lowest available temperature were absolute zero there would be no dissipation of energy.

309. **Increase of Entropy.**—If an element of heat dQ pass from a body A, whose temperature is T_1, to another body B at a lower temperature T_2, and if we suppose the volumes of A and B to remain constant, the entropy of A is diminished by $\frac{dQ}{T_1}$, and that of B increased by $\frac{dQ}{T_2}$, and as $T_1 > T_2$, the whole entropy of A and B is increased.

Again, in a cyclical process, if we suppose the source A and the condenser B to remain at constant volume, in which case their temperatures will of course vary, $\Sigma \frac{Q_1}{T_1}$ is the loss of entropy by A, and $\Sigma \frac{Q_2}{T_2}$ the gain of entropy by B. Hence the entropy of the whole system is increased by the quantity $\Sigma \left(\frac{Q_2}{T_2} - \frac{Q_1}{T_1}\right)$. In a reversible process this quantity is zero, but in a non-reversible process it has in general a positive value N.

We have supposed A and B to remain at constant volume; but if this be not the case, the results obtained still hold good, provided the transformation applied to each of these bodies is reversible when each body is considered alone. Under these circumstances the uncompensated loss of available energy in a non-reversible cyclical process is equal to the product of the limiting temperature and the increase of the entropy of the system.

We may conclude from what has been said, that natural processes have a tendency to increase entropy, or, as stated by Clausius, the entropy of the universe tends to become a maximum.

310. **Path of Least Heat.**—Let us suppose that a body, whose entropy is ϕ_1, passes from the state A to the state B in which its entropy is ϕ_0, less than ϕ_1. If Q be the heat given out by the body when at the temperature T, and if S denote the value of $\Sigma\frac{Q}{T}$ for the whole process, S cannot be less than $\phi_1 - \phi_0$. To prove this, first suppose the transformation reversible, then $S = \phi_1 - \phi_0$. Next suppose the transformation non-reversible, and let the cycle be completed by a reversible process which brings the body from B to A. The value of $\Sigma\frac{Q}{T}$ for the cycle is then $S - (\phi_1 - \phi_0)$, and this must be positive (Art. 308); hence $S > \phi_1 - \phi_0$.

Let us now consider by what path a body, whose temperature can never be less than T_0, should pass from the state A to the state B at T_0, ϕ_0, so that the heat given out in the passage should be a minimum, no heat being supplied to the body from any external source.

Let H be the heat given out; then for a non-reversible transformation, since $T > T_0$, and since any element of heat which enters the body at T must have previously passed out of it at a temperature higher than T, we must have $H > T_0 S > T_0(\phi_1 - \phi_0)$. For a reversible transformation $H = \int_{\phi_0}^{\phi_1} T d\phi$, which is least when $T = T_0$. The least value of H is therefore $T_0(\phi_1 - \phi_0)$. Hence the path consists of an adiabatic at the entropy ϕ_1 from T_1 to T_0, and an isothermal at T_0 from ϕ_1 to ϕ_0. Since $U_1 - U_0 = W + H$, where W is the work done by the body during the transformation, when H is least W is greatest, and the maximum work which a body can perform under the circumstances supposed is

$$U_1 - U_0 - T_0(\phi_1 - \phi_0).$$

EXAMPLES.

1. Prove that the available energy of any system of bodies is

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 \frac{T - T_0}{T} dT,$$

where T_1 is the initial temperature of m_1, and c_1 its specific heat at constant volume.

2. If the system in Ex. 1 be enclosed in an envelope impermeable by heat, show that T_0 is determined by the equation

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 \frac{dT}{T} = 0.$$

The actual work performed by the system during the transformation in which all its parts are brought to the temperature T_0 is

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 \, dT;$$

but, if the transformation be that in which the greatest possible work is done, this work must be equal to the available energy, and therefore

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 \frac{dT}{T} = 0.$$

When the limiting temperature T_0 is determined from within, as in this example, or, in other words, when one part of the system acts as condenser to another part, the available energy is called by Thomson the *Internal Thermodynamic Motivity*. When T_0 is independent of the system, *i.e.* when heat can pass out of the system to an external condenser, the available energy may be termed the *External Thermodynamic Motivity*. In this case T_0 must be assigned.

3. If a system consist of two equal masses of the same substance whose specific heat is constant, show that the limiting temperature of the internal thermodynamic motivity is $\sqrt{T_1 T_2}$, where T_1 and T_2 are the initial temperatures of the two masses.

4. In the preceding example prove that the thermodynamic motivity of the system is $mc(\sqrt{T_1} - \sqrt{T_2})^2$.

5. If the entropy of a substance be increased, its energy remaining constant, prove that the work which can be obtained by a transformation to a given state is diminished.

6. A gas, whose volume is v_1, is allowed to expand into a perfectly empty vessel, whereby its volume becomes v_2; show that its capability of doing work is diminished by the quantity $T_0 R \log \frac{v_2}{v_1}$.

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